STA500 Introduction to Probability and Statistics 2, autumn 2018.

Exercise set 10

Note on Markov processes: Exercise 6, 7, 8, 9.

Exercise 1:

Slippery roads are surveyed. In the winter time there is an inspection at 5 AM each morning to make decisions on how the surface of the road should be treated (sanding, salting, etc). The condition of the road is divided into 3 categories:

- 0: Not slippery
- 1: Snow on the road, somewhat slippery
- 2: Ice on the road, very slippery

To describe the development from one day to another they use a Markov model with transition matrix

$$P = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.6 & 0.1 & 0.3 \end{pmatrix}.$$

a) Comment on how this matrix characterises the condition on the road from one day to another. Find $P(X_{n+1} = 1|X_n = 2)$, $P(X_{n+2} = 0|X_n = 0)$ and $P(X_{n+2} = X_{n+1} = 1|X_n = 0)$.

b) Find the steady state (equilibrium) equations for this system and solve them. In the long run, how much of the time is the road not slippery?

c) The condition of the road is measured in friction units, with 1 friction unit for state 2, 4 for state 1 and 10 for state 0. In terms of these units, find the expected friction. Given that it is not slippery today, what is the expected friction tomorrow? Compare the two answers and comment.

d) In reality the condition on the road does not only depend on the condition of the road yesterday but also on the condition the day before yesterday. Show how you can model this as a 9-state Markov chain. Moreover, set up the transition matrix for this chain when you know the following: Two days in a row with state 0 gives a probability of 0.65 of state 0 the next day, 0.15 of state 1 and 0.2 of state 2. Two days in a row with state 1 or state 2 gives a probability of 0.5 of being in the same state the next day, and 0.25 for each of the other two states. For two days in a row with different states, the state the next day will be equal to last days state with probability 0.5 and equal to the first days state with probability 0.3.

Exercise 2:

The matrix below is the transition probability matrix for a Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$$P = \begin{pmatrix} 0 & 0 & 0.3 & 0 & 0.7 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

a) Draw the transition graph for the Markov chain. Find the classes and decide whether they are transient or recurrent. Find the period for each class. Does this Markov chain have steady state probabilities?

Some answers:

Note on Markov processes, 6: $\pi_0 = 10/19$, $\pi_1 = 4/19$ $\pi_2 = 5/19$.
7 $E(X_t) = 14/19$, $E(X_t|X_{t-2} = 2) = 11/10$.
8 $\pi_0 = (1-p)^2$, $\pi_1 = \pi_2 = (1-p)p$, $\pi_3 = p^2$.
9 $10/13 \approx 0.7692307692$.
1 a) 0.1, 0.63 and 0.1; b) 34/56, 15/56 and 7/56; c) 7.27 and 7.9
2 Classes $\{0, 1, 4, 5\}$, $\{2, 3, 6\}$ and $\{7\}$.