STA500 Introduction to Probability and Statistics 2, autumn 2018.

Exercise set 12

Note on Markov processes: Exercise 14, 15

Exercise 1:

Three high precision machines need to be looked after constantly for them to work perfectly, and they must be taken out and adjusted frequently. Two specialists take care of the adjustments. When a machine is up again, it works satisfactorily for a period of time that is exponentially distributed with a mean of 8 hours. The adjustments of the machines may take quite some time, and this time is exponentially distributed with a mean of 2 hours. Only one specialist can work at a machine at any time. Let the states of this system be the number of machines that are in use, such that the possible states are 0, 1, 2 and 3.

a) Set up the steady state (balance) equations and solve them.

b) In the steady state, find the expected number of machines that are in use and the expected number of specialists that are working. Further, find the proportion of time where at least one machine is up.

Exercise 2:

The telephone queue to a booking office with two operators serving the customers can be modelled as an $M/M/2$ queue with arrival rate $\lambda = 0.4$ per minute and expected service time of $1/\gamma = 4$ minutes. When you call to make a booking you are informed that you are customer number 5 in the queue. This means that two persons are being served and four persons are waiting in front of you in the queue.

a) What is the expected waiting time until you are starting to get served? What is the expected waiting time until you are finished?

b) What is the probability that you have to wait for more than 10 minutes until you start getting served? (Hint: The departure of customers from the queue is also a Poisson process.)

Let $X(t)$ be the number of customers in the system (being served or waiting in queue) at time t.

c) Find the steady state probabilities for $\{X(t) : \; t \geq 0\}$.
(Hint: The following might be useful: $\sum_{k=1}^{\infty} a^k = a/(1-a)$ when $a < 1$.)
Also find the expected number of customers in the system in the long run.
(Hint: The following might be useful: $\sum_{k=1}^{\infty} ka^k = a/(1-a)^2$ when $a < 1$.)

Due to complaints from customers having to wait for a long time in the telephone queue before they are served the booking office decides to only let up to three customers wait in queue. If further customers call when three customers are waiting they are asked to try again later and are not allowed to wait in the queue.

d) Find the steady state probabilities for the number of customers in the system (waiting or being served).
(Hint: The following might be useful: $\sum_{k=1}^{m} a^k = a(1-a^m)/(1-a)$.)
What is the long run proportion of time when new customers are asked to try again later. What is the exepected number of customers in the system now?

Exercise 3:

A steering system has a component A for which the lifetime is exponentially distributed with expectation 30 days. The repair time is also exponentially distributed and with expectation 1 day. If A needs to be repaired, there is a spare component B which takes over for A. Each time it is in use B has a lifetime which is exponentially distributed with expectation 7 days. Component B is only used when A is being repaired. If also B fails, then repair of A is first finished (there is only one repair person) before repair of B starts. If A fails while B is under repair, A is repaired first. The repair time for B is exponentially distributed with expectation 2 days. We introduce the following states: 0: A is functioning and B is not in use, 1: A is under repair and B is in use, 2: A and B are both not functioning, A is being repaired, 3: A is functioning and B is being repaired.

a) In this point neglect the repair time of A and component B, such that the number of times, $N(t)$, which A fails during the time t is a Poisson process where the time T_i between event $i-1$ and i is exponentially distributed with expectation 30 days. Further let $S_i = \sum_{j=1}^{i} T_j$.
Find $P(T_i > 35)$, $P(T_i > 35 | T_i > 25)$, $P(N(30) = 2)$ and $E(S_{10})$.

In the rest of the exercise we take info account the repair time of A and the spare component B as described above.

b) Draw the transition graph and add all the rates.
Set up the steady state equations and solve them.

c) How much of the time in the long run is the system not working?
What is the expected number of components under repair?
How much of the time in the long run is component A under repair?
(If you did not manage to find the steady state probabilities in b), set up a guess of what these probabilities might be and use this guess for solving point c).)

Some answers:

1 a) $\pi_0 = 3/251, \pi_1 = 24/251$, $\pi_2 = 96/251$, $\pi_3 = 128/251$, b) 2.39, 0.60 and 248/251=0.988
2 a) 10 and 14; b) 0.44; c) $\pi_0 = 1/9$, $\pi_k = (2/9)(4/5)^k$ for $k = 1, 2, 3, \ldots$, expectation 40/9; d) 0.16, 0.25, 0.20, 0.16, 0.13, 0.10, expected 2.15
3 a) 0.311, 0.717, 0.184, 300, b) 3600/3751, 105/3751, 16/3751 and 30/3751 c) 0.004, 0.045, 0.032