STA500 Introduction to Probability and Statistics 2, autumn 2018.

Exercise set 4

Textbook: Exercises 8.13, 8.14, 8.21, 8.26, 9.81*, 9.84 (9.82 in 8. edition).

$*$) That $x_1, \ldots, x_n$ are outcomes of a Bernoulli process means that $x_1, \ldots, x_n$ are outcomes of 0-1 variables $X_1, \ldots, X_n$ with distribution $f(x) = p^x(1-p)^{(1-x)} \quad , x = 0, 1$

Exercise 1:

An insurance company assume the payment X after industrial fires to be exponentially distributed. I.e. the pdf of X is:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

The company is in particular interested in the maximum payment, as they consider to reinsure in other companies. Let X_1 and X_2 be two independent payments.

a) Find the probability density of $V = \max(X_1, X_2)$.

b) Find $\mathrm{E}(V)$. Compare with $\mathrm{E}(X)$ and $2\mathrm{E}(X)$ and comment.

Exercise 2:

To find the pH value of a lake with high accuracy a series of measurements has been made. The 10 first measurements, $X_1, \ldots, X_{10}$, was done using an instrument for which it is known that the results of the measurements is having a $N(\mu, 0.01)$ distribution. Here μ is the unknown pH-value we want to estimate.

After 10 measurements was done the instrument did not function anymore, but another 15 measurements $Y_1, \ldots, Y_{15}$ was made using a second instrument for which it is known that the measurements is having a $N(\mu, 0.09)$ distribution.

Afterwards one was wondering how the results of the two measurement series should be combined to a best possible estimator of μ. Two estimators was suggested:

$$\hat{\mu}_1 = \frac{1}{25}\left(\sum_{i=1}^{10} X_i + \sum_{i=1}^{15} Y_i\right) \qquad \text{and} \qquad \hat{\mu}_2 = \frac{6}{7}\bar{X} + \frac{1}{7}\bar{Y}$$

a) Show that both estimators are unbiased. Calculate the variance of both estimators. Which one is the best estimator?

Are there even better estimators?

b) Consider the estimator $\hat{\mu} = a\bar{X} + b\bar{Y}$, and find the values a and b that give the best possible unbiased estimator.

Execise 3:

Two different methods are used for measuring the oil concentration in the water close to an oil-rig. Let μ denote the true oil concentration in the water (measured in suitable units). It is known that the first method gives measurement results which are normally distributed with expectation μ and standard deviation σ_1, and the second method gives measurement results which are normally distributed with expectation μ and standard deviation σ_2.

a) If $\mu = 4.3$ and $\sigma_1 = 0.5$, what is the probability that the first measurement method gives a result below 4.0?
What is the probability that the second measurement method gives a result larger than $\mu + \sigma_2$?

Let X_1 denote the result of a measurement using method 1, and X_2 denote the result of a measurement using method 2. We want to combine the results of two such measurements to a common estimator for μ. Assume that X_1 and X_2 are independent and that σ_1 and σ_2 are known.

b) Show that the maximum likelihood estimator (MLE) for μ based on the two measurements is

$$\hat{\mu} = \frac{\sigma_2^2 X_1 + \sigma_1^2 X_2}{\sigma_1^2 + \sigma_2^2}$$

c) Check if the estimator $\hat{\mu}$ is unbiased.
Show that the variance of the estimator can be expressed: $\text{Var}(\hat{\mu}) = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$

Some answers:

8.13: 0.585; **8.21:** Yes; **8.26:**: a) 0.0062, b) 0.0668, c) 0.3413 ; **9.81:** $\sum_{i=1}^n X_i/n$

9.84/9.82: a) $(\alpha\beta)^n e^{-\alpha \sum_{i=1}^n x_i^\beta} \prod_{i=1}^n x_i^{\beta-1}$; b) $n/\alpha - \sum_{i=1}^n x_i^\beta = 0$

$$n/\beta - \alpha \sum_{i=1}^n x_i^\beta \ln(x_i) + \sum_{i=1}^n \ln(x_i) = 0$$

1: a) $f(v) = 2\lambda(e^{-\lambda v} - e^{-2\lambda v})$, b) $\text{E}(V) = 3/(2\lambda)$

2: a) 0.00232 and 0.000857, i.e. $\hat{\mu}_2$ is best; b) $a = 6/7$ and $b = 1/7$;

3: a) 0.2743 and 0.1587