Solution exam Desember 1, 2014

Problem 1:

a) Integrating the general Weibull density $f(x) = \alpha\beta x^{\beta-1}e^{-\alpha x^\beta}$ and inserting $\alpha = 0.25$ and $\beta = 0.5$ we get:

$$\begin{aligned} F_X(x) &= P(X \le x) = \int_0^x f(u)du = \int_0^x u^{\beta-1}e^{-\alpha u^\beta}du = [-e^{-\alpha u^\beta}]_0^x = 1 - e^{-\alpha x^\beta} = \underline{\underline{1 - e^{-0.25x^{0.5}}}} \\ P(X > 3) &= 1 - P(X \le 3) = 1 - (1 - e^{-0.25\cdot 3^{0.5}}) = e^{-0.25\cdot 3^{0.5}} = \underline{\underline{0.65}} \end{aligned}$$

The hazard rate becomes:

$$r(x) = \frac{f(x)}{1 - F(x)} = \frac{0.25 \cdot 0.5x^{0.5-1}e^{-0.25x^{0.5}}}{e^{-0.25t^{0.5}}} = \underline{\underline{0.125x^{-0.5}}}$$

We see that the hazard rate is decreasing with time - i.e. the older the components get the less likely they are to fail in the near future.

$$P(X > 5|X > 2) = \frac{P(X > 5 \cap X > 2)}{P(X > 2)} = \frac{P(X > 5)}{P(X > 2)} = \frac{e^{-0.25\cdot 5^{0.5}}}{e^{-0.25\cdot 2^{0.5}}} = \underline{\underline{0.81}}$$

b) We have a situation where

- We check "success" or not "success" in each trial - whether the CPU in a robot fails within 3 years or not.
- The probability of "success" is the same in all trials, $p = P(X < 3) = 1 - 0.65 = 0.35$.
- Independent trials - independent between robots whether CPU fails within 3 years or not.
- We have a specified number of trials - 52 robots.

Then Y has a $\underline{\underline{\text{binomial distribution with } n = 52 \text{ and } p = 0.35}}$.

$$\begin{aligned} \mathrm{E}(Y) &= np = 52 \cdot 0.35 = \underline{\underline{18.2}} \\ \mathrm{Var}(Y) &= np(1-p) = 52 \cdot 0.35 \cdot 0.65 = \underline{\underline{11.83}} \\ P(Y > 20) &= 1 - P(Y \le 20) \approx 1 - P(Z \le \frac{20 + 0.5 - \mathrm{E}(Y)}{\sqrt{\mathrm{Var}(Y)}}) \\ &= 1 - P(Z \le \frac{20 + 0.5 - 18.2}{\sqrt{11.83}}) = 1 - P(Z \le 0.67) = 1 - 0.7482 = \underline{\underline{0.25}} \end{aligned}$$

We can use the approximation to the normal distribution since $\mathrm{Var}(Y) > 5$. Without the correction factor $+0.5$ the answer becomes 0.30.

c)
$$\begin{aligned} F_U(u) &= P(U \le u) = P(\min(X_1, X_2, X_3) \le u) = 1 - P(\min(X_1, X_2, X_3) > u) \\ &\overset{indep.}{=} 1 - P(X_1 > u) \cdot P(X_2 > u) \cdot P(X_3 > u) = 1 - [1 - F_X(u)]^3 \\ &= 1 - [e^{-0.25u^{0.5}}]^3 = \underline{\underline{1 - e^{-0.75u^{0.5}}}} \\ P(U > 3) &= 1 - P(U \le 3) = 1 - (1 - e^{-0.75 3^{0.5}}) = e^{-0.75 3^{0.5}} = \underline{\underline{0.27}} \end{aligned}$$

If we compare the cdf of U above with the general expression for the cdf of a Weibull distribution, derived in the first problem in a), we see that $F_U(u)$ is on the form of a Weibull distribution and with parameter values $\alpha = 0.75$ and $\beta = 0.5$. (Or we can find the same by taking the derivative of $F_U(u)$ and compare to the Weibull density.) Then using the expression for the expectation in the Weibull distribution:

$$\begin{aligned} \mathrm{E}(U) &= \alpha^{-1/\beta}\Gamma(1 + \frac{1}{\beta}) = 0.75^{-1/0.5}\Gamma(1 + \frac{1}{0.5}) = 0.25^{-2}\Gamma(3) = 0.75^{-2} \cdot 2! = \underline{\underline{3.55}} \\ E(X) &= \alpha^{-1/\beta}\Gamma(1 + \frac{1}{\beta}) = 0.25^{-1/0.5}\Gamma(1 + \frac{1}{0.5}) = 0.25^{-2}\Gamma(3) = 0.25^{-2} \cdot 2! = \underline{\underline{32}} \end{aligned}$$

We see that $\mathrm{E}(U)$ is much lower than $\mathrm{E}(X)$ (a factor 9 lower). With a strongly decreasing failure rate many units experience an early failure, but some live for a very long time and the expected lifetime is quite high. However, with three components that could fail we will often see an early failure in at least one of them and rarely see a long lifetime of the system, giving a low expected time to failure for the system.

Exercise 2:

a)

$$\begin{aligned} P(X_2 = 2|X_1 = 1) &= \underline{\underline{0.12}} \\ P(X_5 = 3|X_4 = 2, X_3 = 1, X_2 = 1, X_1 = 1) &= P(X_5 = 3|X_4 = 2) = \underline{\underline{0.05}} \\ P(X_4 = 2, X_3 = 1, X_2 = 1|X_1 = 1) &= p_{11}p_{11}p_{12} = 0.83 \cdot 0.83 \cdot 0.12 = \underline{\underline{0.083}} \\ P(X_2 \ne 2|X_1 = 2) &= 1 - P(X_2 = 2|X_1 = 2) = 1 - 0.85 = \underline{\underline{0.15}} \end{aligned}$$

b) The number of steps until we leave state 2 has a geometric distribution (independent from step to step if we go to a new state or not, same probability of going to a new state in each step, we record whether we go to a new state or not and we record the number of steps until first time we leave the state) with probability $p = 0.15$ of leaving the state. The expected number of months until we leave the state is thus $1/p = 1/0.15 = \underline{\underline{6.7}}$. The probability for staying in the state is highest for state 0, and the expected number of months until the process go to some other state is thus highest for $\underline{\underline{\text{state } 0}}$. The expectation is $1/0.07 = 14.3$.

The Markov chain is irreducible (just one class, all states communicate), posivite recurrent (guaranteed to return to all states in a finite number of steps) and aperiodic (period 1) - i.e. the Markov chain has steady state probabilities.

The P^{12}-matrix give probabilities for were the process is 12 steps into the future for all possible starting points. I.e. in this specific case the matrix gives the probabilities for which state the oil price will be in one year into the future for all possible current states.

$$P(X_{n+12} > 2|X_n = 2) = P(X_{n+12} = 3|X_n = 2) + P(X_{n+12} = 4|X_n = 2) = 0.25 + 0.04 = \underline{\underline{0.29}}$$

The interpretation of this is that if the price a month is instate 2, the probability that the price one year later will be in a higher state is 0.29.

c)

$$\begin{pmatrix}\pi_0\\ \pi_1\end{pmatrix} = \begin{pmatrix}0.38 & 0.42\\ 0.62 & 0.58\end{pmatrix}\begin{pmatrix}\pi_0\\ \pi_1\end{pmatrix}$$

which gives

$$\begin{aligned}\pi_0 &= 0.38\pi_0 + 0.42\pi_1\\ \pi_1 &= 0.62\pi_0 + 0.58\pi_1\end{aligned}$$

By choosing for instance the first of these equations and $\pi_0 + \pi_1 = 1$ and solving these two equations we get $\pi_0 = \underline{\underline{0.40}}$ and $\pi_1 = \underline{\underline{0.60}}$.

The steady state probabilities give respectively the proportion of months when the price is down and the proportion of months when the price is up in the long run. We see that the price more often go up than down.

We can define new states holding information about the state the previous month and this month e.g. as, 0: "down, down", 1: "up, down", 2: "down, up", 3: "up, up". Since each state holds information about the price the two most recent months this will be a Markov chain when the memory of the process goes two months back. The transition matrix becomes:

$$P = \begin{pmatrix} x & 0 & x & 0\\ x & 0 & x & 0\\ 0 & x & 0 & x\\ 0 & x & 0 & x\end{pmatrix}$$

Exercise 3:

a)

$$\begin{aligned}\mathrm{E}(\hat\lambda_1) &= \mathrm{E}\Big(\frac{1}{n}\sum_{i=1}^{n}\frac{X_i}{t_i}\Big) = \frac{1}{n}\sum_{i=1}^{n}\frac{\mathrm{E}(X_i)}{t_i} = \frac{1}{n}\sum_{i=1}^{n}\frac{\lambda t_i}{t_i} = \frac{1}{n}\sum_{i=1}^{n}\lambda = \underline{\underline{\lambda}}\\ \mathrm{Var}(\hat\lambda_1) &= \mathrm{Var}\Big(\frac{1}{n}\sum_{i=1}^{n}\frac{X_i}{t_i}\Big) \overset{indep.}{=} \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}\Big(\frac{X_i}{t_i}\Big)\\ &= \frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{t_i^2}\mathrm{Var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{t_i^2}\lambda t_i = \underline{\underline{\frac{\lambda}{n^2}\sum_{i=1}^{n}\frac{1}{t_i}}}\end{aligned}$$

Estimates:

$$\hat\lambda_1 = \frac{1}{5}\sum_{i=1}^{5}\frac{x_i}{t_i} = \frac{1}{5}0.043 = \underline{\underline{0.0086}}, \qquad \hat\lambda_2 = \frac{\sum_{i=1}^{5}x_i}{\sum_{i=1}^{5}t_i} = \frac{8}{900} = \underline{\underline{0.0089}}$$

Since both estimators are unbiased, we prefer the estimator with lowest variance. From the expression above we get:

$$\begin{aligned}\mathrm{Var}(\hat\lambda_1) &= \frac{\lambda}{5^2}\sum_{i=1}^{5}\frac{1}{t_i} = \frac{\lambda}{25}\cdot 0.041 = 0.00164\cdot\lambda\\ \mathrm{Var}(\hat\lambda_2) &= \frac{\lambda}{\sum_{i=1}^{5}t_i} = \frac{\lambda}{900} = 0.0011\cdot\lambda\end{aligned}$$

i.e. we see that $\hat\lambda_2$ has lowest variance and we will thus $\underline{\underline{\text{prefer } \hat\lambda_2.}}$

b)

$$\begin{aligned}L(\lambda) &= \prod_{i=1}^{n}f(x_i;\lambda) = \prod_{i=1}^{n}\frac{(\lambda t_i)^{x_i}}{x_i!}e^{-\lambda t_i} = \frac{\prod_{i=1}^{n}(\lambda t_i)^{x_i}}{\prod_{i=1}^{n}x_i!}e^{-\lambda\sum_{i=1}^{n}t_i} = \frac{\lambda^{\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}t_i^{x_i}}{\prod_{i=1}^{n}x_i!}e^{-\lambda\sum_{i=1}^{n}t_i}\\ \ln(L(\lambda)) &= \sum_{i=1}^{n}x_i\ln(\lambda) + \ln\Big(\prod_{i=1}^{n}t_i^{x_i}\Big) - \ln\Big(\prod_{i=1}^{n}x_i!\Big) - \lambda\sum_{i=1}^{n}t_i\\ \frac{\partial\ln L(\lambda)}{\partial\lambda} &= \frac{\sum_{i=1}^{n}x_i}{\lambda} - \sum_{i=1}^{n}t_i = \frac{1}{\lambda}\sum_{i=1}^{n}x_i - \sum_{i=1}^{n}t_i = 0 \qquad\Rightarrow\qquad \underline{\underline{\hat\lambda = \frac{\sum_{i=1}^{n}X_i}{\sum_{i=1}^{n}t_i}}} = \hat\lambda_2\end{aligned}$$

To find the Wald confidence interval we start by finding the second derivative of the log-likelihood at $\hat\lambda$:

$$\begin{aligned}J(\lambda) &= \frac{\partial^2\ln L(\lambda)}{\partial\lambda^2} = -\frac{1}{\lambda^2}\sum_{i=1}^{n}x_i\\ J(\hat\lambda) &= -\frac{1}{\hat\lambda^2}\sum_{i=1}^{n}X_i = -\frac{\sum_{i=1}^{n}X_i}{\Big(\frac{\sum_{i=1}^{n}x_i}{\sum_{i=1}^{n}t_i}\Big)^2} = \frac{1}{\frac{\sum_{i=1}^{n}x_i}{(\sum_{i=1}^{n}t_i)^2}} = -\frac{\sum_{i=1}^{n}t_i}{\hat\lambda}\end{aligned}$$

The Wald confidence interval then becomes:

$$[\hat\lambda - z_{\alpha/2}\sqrt{-1/J(\hat\lambda)}, \hat\lambda + z_{\alpha/2}\sqrt{-1/J(\hat\lambda)}] = \underline{\underline{[\hat\lambda - z_{\alpha/2}\sqrt{\hat\lambda/\sum_{i=1}^{n}t_i}, \hat\lambda + z_{\alpha/2}\sqrt{\hat\lambda/\sum_{i=1}^{n}t_i}]}}$$

With $\sum_{i=1}^{12}X_i = 25$ and $\sum_{i=1}^{12}t_i = 3200$ we get $\hat\lambda = 25/3200 = 0.0078$. Also $z_{\alpha/2} = z_{0.05} = 1.645$ and we get the interval:

$$[0.0078 - 1.645\sqrt{0.0078/3200},\ \ 0.0078 + 1.645\sqrt{0.0078/3200}] = \underline{\underline{[0.005,\ \ 0.010]}}$$

The estimator $\hat\lambda = \sum_{i=1}^{n}X_i/\sum_{i=1}^{n}t_i$ is approximately normally distribution when $\sum_{i=1}^{n}X_i$ is approximately normally distributed. $\sum_{i=1}^{n}X_i$ is Poisson-distributed with expectation $\lambda\sum_{i=1}^{n}t_i$ and is thus well approximated with a normal distribution when $\lambda\sum_{i=1}^{n}t_i = \lambda\cdot 3200 > 15$. This holds for all values of λ in the confidence interval and we thus conclude that we can thrust the interval.

c) With the likelihood from b) and the gamma prior distribution we get the posterior distribution:

$$\begin{aligned}p(\lambda|x_1,\ldots,x_n) &= c\cdot L(\lambda)\cdot p(\lambda) = c\cdot\frac{\lambda^{\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}t_i^{x_i}}{\prod_{i=1}^{n}x_i!}e^{-\lambda\sum_{i=1}^{n}t_i}\frac{1}{b^a\Gamma(a)}\lambda^{a-1}e^{-\lambda/b}\\ &= c_2\cdot\lambda^{\sum_{i=1}^{n}x_i+a-1}e^{-\lambda(\sum_{i=1}^{n}t_i+1/b)}\end{aligned}$$

Comparing with the gamma density we see that this posterior distribution is a gamma distribution with parameters $\sum_{i=1}^{n}x_i + a$ and $1/(\sum_{i=1}^{n}t_i + 1/b)$. The Bayes estimate is then the expectation of this gamma distribution which is the product of the two parameters:

$$\hat\lambda_{\text{Bayes}} = \underline{\underline{\frac{\sum_{i=1}^{n}x_i + a}{\sum_{i=1}^{n}t_i + 1/b}}}$$

To calculate the estimate numerically we first we have to find the values of a and b. From the information in the text we have that the expectation in the prior distribution is $ab = 0.01$ and variance $ab^2 = 0.00001$. Inserting the first in the latter we get $0.01b = 0.00001$ which implies $b = 0.001$ and thus $a = 10$. Using this and the data information given before point b) we get:

$$\hat\lambda_{\text{Bayes}} = \frac{25+10}{3200+1/0.001} = \underline{\underline{0.0083}}$$

The maximum likelihood estimate is 0.0078 and the prior expectation is 0.01. The Bayes estimate 0.0083 is much closer to the data estimate than the prior estimate, i.e. the data information is given much more weight than the prior information.