Solution exercise set 10

Note on Markov processes, Exercise 6

The process admit steady state probabilities as it has finite state space, is irreducible (all states can be reached from any other state in one transition) and is aperiodic (all diagonal elements in P are > 0). The steady state probabilities are found by solving two out of

$$\pi_0 = 4/5\pi_0 + 1/4\pi_1 + 1/5\pi_2$$
$$\pi_1 = 1/10\pi_0 + 1/2\pi_1 + 1/5\pi_2$$
$$\pi_2 = 1/10\pi_0 + 1/4\pi_1 + 3/5\pi_2$$

along with $\pi_0 + \pi_1 + \pi_2 = 1$. This results in $\pi_0 = 10/19,\ \pi_1 = 4/19\ \pi_2 = 5/19$.

Note on Markov processes, Exercise 7

The two period transition probability matrix obtains as $P^2 = PP$. E.g. the first diagonal then becomes

$$p_{00}^2 = \sum_{k=0}^{2} p_{0k}p_{k0} = (4/5)^2 + (1/10)(1/4) + (1/10)(1/5) = 137/200$$

and so on for the remaining elements. The two period transition probability $P(X_t = 0|X_{t-2} = 2) = p_{20}^2 = 33/100$. The unconditional expectation of the process is

$$E(X_t) = 0\pi_0 + 1\pi_1 + 2\pi_2 = 14/19 \approx 0.7368421053.$$

The conditional expectation

$$E(X_t|X_{t-2} = 2) = 0p_{20}^2 + 1p_{21}^2 + 2p_{22}^2 = 11/10 = 1.1$$

We see that the conditional expectation is slightly higher than the unconditional one, which is a consequence of the fact that the process has a quite high probability of remaining in state 2 for two transitions.

Note on Markov processes, Exercise 8

The process X_t is a Markov chain as there is no additional information about X_t (depending on Z_t, Z_{t-1} in knowing X_{t-2} (depending on Z_{t-2}, Z_{t-3}) or any other $X_{t-k},\ k = 2, \dots$.

The process admit the following transitions

- $X_{t-1} = 0 \rightarrow Z_{t-1} = 0:$ Transitions to 0 ($Z_t = 0$, prob=$1-p$) or 1 ($Z_t = 1$, prob=p) possible
- $X_{t-1} = 1 \rightarrow Z_{t-1} = 1:$ Transitions to 2 ($Z_t = 0$, prob=$1-p$) or 3 ($Z_t = 1$, prob=p) possible.

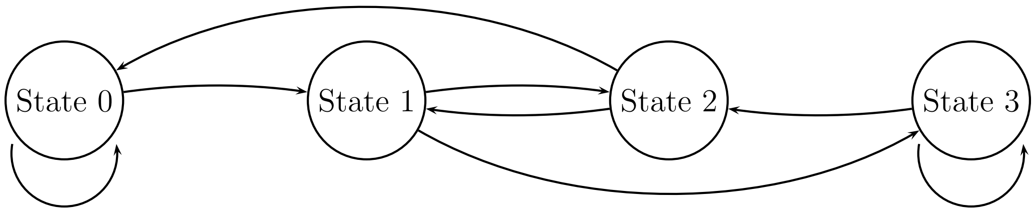
- $X_{t-1} = 2 \to Z_{t-1} = 0:$ Transitions to 0 ($Z_t = 0$, prob=$1-p$) or 1 ($Z_t = 1$, prob=p) possible.

- $X_{t-1} = 3 \to Z_{t-1} = 1:$ Transitions to 2 ($Z_t = 0$, prob=$1-p$) or 3 ($Z_t = 1$, prob=p) possible.

This gives us the transition probability matrix

$$P = \begin{pmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \\ 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \end{pmatrix}.$$

In order to determine whether X_t is irreducible and aperiodic, it is helpful to draw the transition graph of the process (for simplicity, skipping labeling the arcs):



The process is irreducible (all states are reachable from every other in two transitions), and the single class is aperiodic as e.g. $p_{00} > 0$.

Looking carefully at the definition of the process, we see that X_t (which depend on Z_t, Z_{t-1}) and X_{t+2} (which depend on Z_{t+2}, Z_{t+1}) are independent in this case, and therefore

$$P(X_{t+2} = i | X_t = j) = P(X_{t+2} = i) = \pi_i.$$

Thus the steady state probabilities are easily found by calculating any row of P^2, e.g. here we compute the first row:

$$\pi_0 = p_{00}^2 = (1-p)^2 + p \times 0 + 0 \times (1-p) + 0 \times 0 = (1-p)^2,$$

and similiarly:

$$\pi_1 = p_{01}^2 = p(1-p),\ \pi_2 = p_{02}^2 = p(1-p),\ \pi_3 = p_{03}^2 = p^2.$$

Alternatively, one could have found the steady state probabilities by directly reasoning from the definition of X_t, e.g. $\pi_0 = P(X_t = 0) = P(Z_t = 0, Z_{t-1} = 0) = P(Z_t = 0)P(Z_{t-1} = 0) = (1-p)^2$ and so on.

Note on Markov processes, Exercise 9

We let X_t be the markov chain so that $X_t = 0$ correspond to salmon t being of grade A and $X_t = 1$ correspond to salmon t being of grade B. From the exercise, we have that $P(X_{t+1} = 0 | X_t = 0) = 0.97 = p_{00}$ and $P(X_{t+1} = 0 | X_t = 1) = 0.1 = p_{10}$. This is sufficient information to find the complete transtion probability matrix of X_t (as the rows of P must sum to 1):

$$P = \begin{pmatrix} 0.97 & 0.03 \\ 0.1 & 0.9 \end{pmatrix}$$

To find the fraction of grade A salmon, which is equal to π_0, we solve (using first steady state equation)

$$\pi_0 = 0.97\pi_0 + 0.1\pi_1,$$

$$\pi_0 + \pi_1 = 1,$$

Standard calculations lead to $\pi_0 = 10/13, \pi_1 = 3/13$, thus the fraction of grade A salmon is $10/13 \approx 0.769$, i.e. approximately 76.9 %.

Exercise 1:

a) We read directly from the transition probability matrix that $P(X_{n+1} = 1|X_n = 2) = p_{21} = \underline{\underline{0.1}}$. To find $P(X_{n+2} = 0|X_n = 0)$ we can either consider all possible two step transition from 0 to 0. Or we can calculate P^2 and read out directly (it is actually sufficient to calculate the first element of the matrix):

$$P^2 = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.6 & 0.1 & 0.3 \end{pmatrix} \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.6 & 0.1 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.63 & 0.25 & 0.12 \\ 0.54 & 0.34 & 0.12 \\ 0.64 & 0.20 & 0.15 \end{pmatrix}$$

From this we see that $P(X_{n+2} = 0|X_n = 0) = p^2_{00} = \underline{\underline{0.63}}$. Finally $P(X_{n+2} = X_{n+1} = 1|X_n = 0) = P(X_{n+2} = 1|X_{n+1} = 1)P(X_{n+1} = 1|X_n = 0) = p_{01}p_{11} = 0.5 \cdot 0.2 = \underline{\underline{0.1}}$.

b) The steady state equations becomes:

$$0.7\pi_0 + 0.4\pi_1 + 0.6\pi_2 = \pi_0$$

$$0.2\pi_0 + 0.5\pi_1 + 0.1\pi_2 = \pi_1$$

$$0.1\pi_0 + 0.1\pi_1 + 0.3\pi_2 = \pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

Solving these gives $\pi_0 = \underline{\underline{34/56}}$, $\pi_1 = \underline{\underline{15/56}}$,$\pi_2 = \underline{\underline{7/56}}$. Not slippery 34/56 parts of the time.

c) We find the expected friction by summing all friction numbers multiplyed by their probability.

Expected friction in steady state: $\text{E(Friction)} = 10 \cdot \frac{34}{56} + 4 \cdot \frac{15}{56} + 1 \cdot \frac{7}{56} = \underline{\underline{\frac{407}{56} = 7.27}}$.

$$\text{E(Friction tomorrow|not slippery today)} = 10 \cdot 0.7 + 4 \cdot 0.2 + 1 \cdot 0.1 = \underline{\underline{7.9}} > 7.27,$$

which is reasonable as one would expect it to be less slippery tomorrow when we know that it is not slippery today.

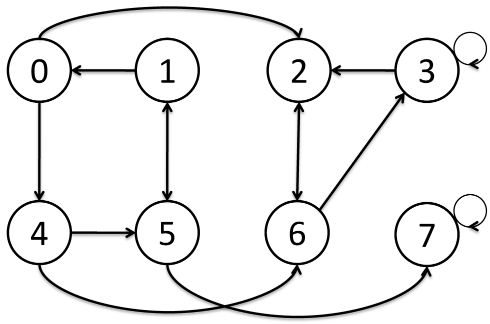
d) Let "n" denote not slippery, "s" denote snow and "i" denote ice. Then we can e.g. label the states (yesterday, today): 0 = nn, 1 = sn, 2 = in, 3 = ns, 4 = ss, 5 = is, 6 = ni, 7 = si, 8 = ii With $\{X_n : n = 0, 1, 2, \dots\}$ defined on this state space we have a Markov chain since X_n carry information about the condition the two previous days and X_{n+1}

will thus only depend on X_n. From the information in the text we get:

$$P = \begin{pmatrix} 0.65 & 0 & 0 & 0.15 & 0 & 0 & 0.20 & 0 & 0 \\ 0.5 & 0 & 0 & 0.3 & 0 & 0 & 0.2 & 0 & 0 \\ 0.5 & 0 & 0 & 0.2 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0.5 & 0 & 0 & 0.2 & 0 \\ 0 & 0.25 & 0 & 0 & 0.5 & 0 & 0 & 0.25 & 0 \\ 0 & 0.2 & 0 & 0 & 0.5 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0.2 & 0 & 0 & 0.5 \\ 0 & 0 & 0.2 & 0 & 0 & 0.3 & 0 & 0 & 0.5 \\ 0 & 0 & 0.25 & 0 & 0 & 0.25 & 0 & 0 & 0.5 \end{pmatrix}$$

Exercise 2:

a) The transition graph is displayed in the figure below. From the transition graph we see



that the classes are (remember that in a class all states communicate) $\underline{\underline{\{0,1,4,5\}}}$,$\underline{\underline{\{2,3,6\}}}$ and $\underline{\underline{\{7\}}}$.

Further we see that if we are in states in the first class we are not guaranteed to return, i.e. $\underline{\underline{\{0,1,4,5\}\text{ is transient}}}$. The classes $\{2,3,6\}$ and $\{7\}$ are absorbing, if we start in one of the states in one of these classes we remain in the state forever and are guaranteed to come back to the state we start in (an infinite number of times), i.e. classes $\underline{\underline{\{2,3,6\}\text{ and }\{7\}\text{ are recurrent}}}$.

For the states in the class $\{0,1,4,5\}$ we see that we can return in $\{2,4,6,8,\ldots\}$ or $\{4,6,8,\ldots\}$ steps. The greatest common divisor is 2, i.e. the class $\{0,1,4,5\}$ has $\underline{\underline{\text{period 2}}}$.

For the states in the class $\{2,3,6\}$ we see that we can return in $\{2,3,4,5,\ldots\}$ or in $\{1,2,3,4,\ldots\}$ steps. The greatest common divisor is 1, i.e. the class $\{2,3,6\}$ is having $\underline{\underline{\text{period 1}}}$. The class $\{7\}$ also has $\underline{\underline{\text{period 1}}}$.