STA500 Introduction to Probability and Statistics 2, autumn 2018.

Solution exercise set 11

Note on Markov processes, Exercise 10
The process is irreducible as any state is reachable from any other state in a finite number of transitions. The process is periodic with period 2 as e.g. state 0 can only be re-visited in $2, 4, 6, \ldots$ transitions.

The transition probability matrix is given as

$$P = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

and the two period transition matrix is

$$P^2 = PP = P = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

From these matrices, it is clear that

$$P(X_{t+2} = 0|X_t = 0) = P_{00}^2 = 1/2,$$

$$P(X_{t+3} = 0|X_t = 0) = P_{00}^3 = P_{00} = 0,$$

$$E(X_{t+4}|X_t = 0) = \sum_{i=0}^{3} ip_{0i}^4 = \sum_{i=0}^{3} ip_{0i}^2 = 0 \times 1/2 + 1 \times 0 + 2 \times 1/2 + 3 \times 0 = 1$$

Note on Markov processes, Exercise 11
First question: Firstly, this is a trick question, since we are discussing a Markov chain, so the information on what happened the day before yesterday and yesterday is redundant. Thus the sought probability is

$$P(X_{t+1} \neq 1|X_t = 1) = 1 - P(X_{t+1} = 1|X_t = 1) = 1 - 1/2 = 1/2.$$

Last question: The sought probability is really the probability that either of the following happens: $X_{t+1} = 0, X_{t+2} = 0$ or $X_{t+1} = 2, X_{t+2} = 0$ or $X_{t+1} = 0, X_{t+2} = 2$ or $X_{t+1} = 2, X_{t+2} = 2$, conditionally on $X_t = 1$:

$$P(X_{t+1} \neq 1, X_{t+2} \neq 1|X_t = 1) = p_{10}p_{00} + p_{12}p_{20} + p_{10}p_{02} + p_{12}p_{22} = 17/40.$$

Note on Markov processes, Exercise 12
When we start in state 0, in the next step we either remain in state 0 with a probability of 0.97 or go to state 1 with a probability of 0.03. Let Y denote the number of transitions until the first visit to state 1. The situation is characterised by:

- Independent trials - independent from step to step whether we leave state 0 or not (due to the Markov assumption).

- Record “success”/”not success”- here “success” corresponds to leaving state 0.
- The same probability of success in each trial - same probability $p = 0.03$ of leaving state 0 in the next step each time when we stay in state 0.
- Repeated trials until the first success - record the number of transitions until we leave state 0.

I.e. Y is having a geometric distribution with $p = 0.03$ and then $\mathrm{E}(Y) = 1/p \approx 33.333$.

Note on Markov processes, Exercise 13

The information says that 34 requests occurred (as a Poisson process) during an 8 hour period, and asks what the probability of at least one request occurring in the first 15 minutes (conditional on knowing the above info:

The number of requests X in an interval of length $s = 0.25$ hours when $N(t) = 34$, $t = 8.0$ hours has a binomial $B(34, 0.25/8.0) = B(34, 0.03125)$ distribution. The question translates to $P(X \geq 1) = 1 - P(X = 0) = 1 - (1 - 0.03125)^{34} \approx 0.66$.

Exercise 1:

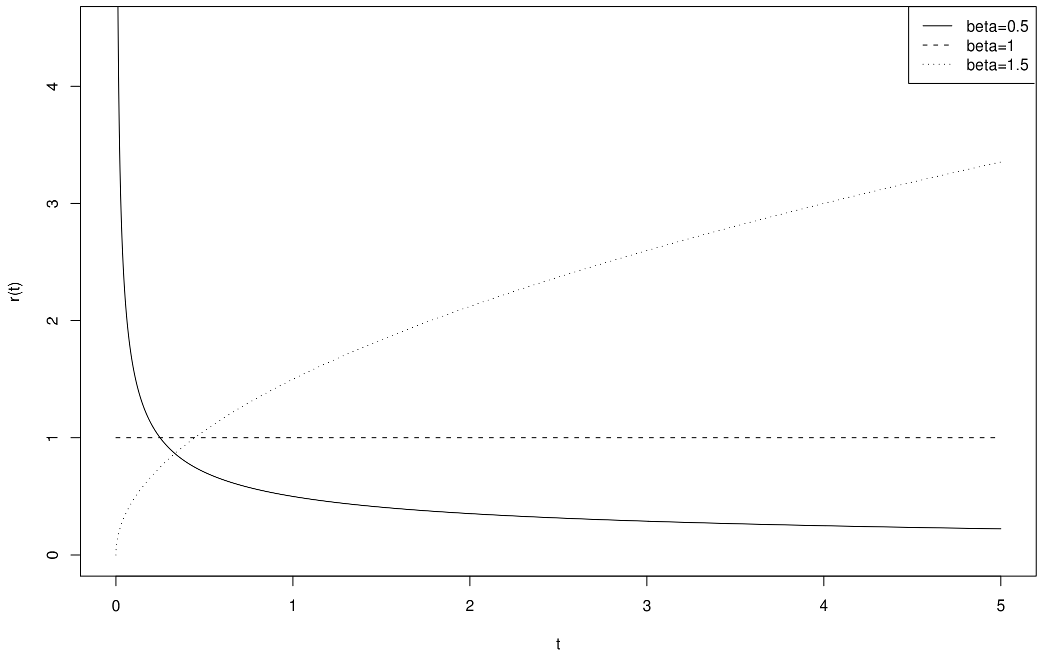
a) For the Weibull distribution we have:

$$F(t) = \int_0^t \alpha\beta u^{\beta-1} e^{-\alpha u^\beta} = [-e^{-\alpha u^\beta}]_0^t = 1 - e^{-\alpha t^\beta}$$

and we then get:

$$r(t) = \frac{f(t)}{1-F(t)} = \frac{\alpha\beta t^{\beta-1} e^{-\alpha t^\beta}}{e^{-\alpha t^\beta}} = \underline{\underline{\alpha\beta t^{\beta-1}}}$$

Plots of the hazard rate for $\alpha = 1$ and $\beta = 0.5$ (decreasing), $\beta = 1$ (constant) and $\beta = 1.5$ (increasing) are displayed in the figure below.



For $\beta = 0.5$ the hazard rate is decreasing, meaning that valves are less likely to fail in the near future the older they get (could mean that the weak vales tend to fail early, those remaining after some time are less likely to fail). For $\beta = 1$ the hazard rate is constant, meaning that valves neither gets more nor less likely to fail in the near future the older they get. For $\beta = 1.5$ the hazard rate is increasing, meaning that valves are more likely to fail in the near future the older they get (ageing).

b)

$$\begin{aligned}
\beta = 0.5: \quad & P(T>2) = 1 - P(T \le 2) = 1 - (1 - e^{-2^{0.5}}) = e^{-2^{0.5}} = \underline{\underline{0.243}} \\
& P(T>4|T>2) = \frac{P(T>4 \cap T>2)}{P(T>2)} = \frac{P(T>4)}{P(T>2)} = \frac{e^{-4^{0.5}}}{e^{-2^{0.5}}} = \underline{\underline{0.557}} \\
\beta = 1: \quad & P(T>2) = e^{-2^1} = \underline{\underline{0.135}} \\
& P(T>4|T>2) = \frac{P(T>4 \cap T>2)}{P(T>2)} = \frac{P(T>4)}{P(T>2)} = \frac{e^{-4^1}}{e^{-2^1}} = \underline{\underline{0.135}} \\
\beta = 1.5: \quad & P(T>2) = e^{-2^{1.5}} = \underline{\underline{0.059}} \\
& P(T>4|T>2) = \frac{P(T>4 \cap T>2)}{P(T>2)} = \frac{P(T>4)}{P(T>2)} = \frac{e^{-4^{1.5}}}{e^{-2^{1.5}}} = \underline{\underline{0.0057}}
\end{aligned}$$

For $\beta = 0.5$ the probability that a two years old valve is functioning more then two new years is larger than the probability that a new valve is functioning more than two years. I.e. the longer a valve has survived, the more likely it is to keep functioning. This is the same property as we saw by the decreasing hazard rate in a).

For $\beta = 1$ the probability that a two years old valve is functioning more then two new years is the same as the probability that a new valve is functioning more than two years. For $\beta = 1$ the Weibull distribution equals the exponential distribution, and this is the memory-less property of the exponential distribution which we also saw by the constant hazard rate in a). I.e. neither improvement nor ageing over time for this case.

For $\beta = 1.5$ the probability that a two years old valve is functioning more then two new years is smaller than the probability that a new valve is functioning more than two years. I.e. the longer a valve has survived, the more likely it is to fail (it is ageing). This is the same property as we saw by the increasing hazard rate in a).

Exercise 2:

a) The queue is stable as long as the rate of transitions in to the queue is smaller than the rate of transition out of the queue. We see that

$$\lambda = 1.5 < c\gamma = 4 \cdot 0.5 = 2$$

i.e. the queue is stable. However, if one of the four persons serving customers have to leave the queue is no longer stable, then $\lambda = 1.5 = 3 \cdot 0.5$.

b) Let $W_{11} = Z_1 + Z_2 + \cdots + Z_{11}$ denote the waiting time until you start to get served. Here Z_i denotes the time from when customer number i in the queue becomes first in line until he is being served. (Except Z_1 which is the time from when you enter until the first customer in line is being served, but due to the memoryless property of the exponential distribution Z_1 is having the same distribution as the other Z_i.)

Another way to view the Z_is is that they are the times between each time a customer leave the shop - and 11 customers have to leave the shop before it is your time to be served.

Further $Z_i = \min\{V_1, V_2, V_3, V_4\}$ where V_j is the time until server j is ready to take a new customer. Due to the memoryless property of the exponential distribution, from when customer i moves up to be first in line $V_1, \dots, V_4$ are iid exponential with $\mathrm{E}(V_j) = 2$. We then know that since $Z_i = \min\{V_1, V_2, V_3, V_4\}$ we have that Z_i is exponentially distributed with $\mathrm{E}(Z_i) = \mathrm{E}(V_j)/4 = 0.5$. Then:

$$\mathrm{E}(W_{11}) = \mathrm{E}(Z_1) + \mathrm{E}(Z_2) + \cdots + \mathrm{E}(Z_{11}) = 11 \cdot 0.5 = \underline{\underline{5.5}}$$

Let C denote the service time. The time until you are finished is then $W_{11} + C$ and

$$\mathrm{E}(W_{11} + C) = \mathrm{E}(W_{11}) + \mathrm{E}(C) = 5.5 + 2 = \underline{\underline{7.5}}$$

c) For you to be finished before the person in front of you, this person needs to still be served when you starts to get served, and then you need to get finished before him. Due to the memoryless property of the exponential distribution for the service times this can be calculated the following way:

$$\begin{aligned}P(\text{you finish before person in front}) &= P(\text{person in front is still being served when you start}) \\ &\quad \times P(\text{you finish first}|\text{person in front is still being served when you start}) \\ &= \tfrac{3}{4} \cdot \tfrac{1}{2} = \underline{\underline{\tfrac{3}{8}}}\end{aligned}$$

When the person in front enter, due to the memoryless property, the distribution of the remaining service time is the same for all four customers, and it is thus the same probability for each of the four to finish first. When you enter, again due to the memoryless property, the distribution of the remaining service time is the same for you and the person which originally was in front of you, and the two of you have the same chance of finishing first.

The probability that the person behind you finish before you is the same as the probability that you finish before the person in front of you, i.e. $\underline{\underline{3/8}}$.

Exercise 3:

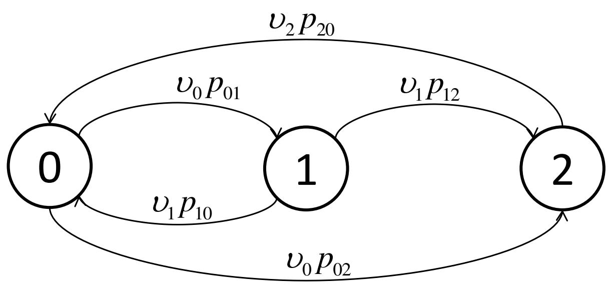
a) In practice the fact that the state of the machine is a continuous time Markov chain means that the state is observed continuously and that only the current state of the machine (and not information about past states) is relevant for predicting the future state development. In mathematical terms, for all $s, t > 0$:

$$P(X(s+t) = j | X(s) = i, X(u) = x_u \text{ for } 0 \leq u < s) = P(X(s+t) = j | X(s) = i)$$

In a continuous time Markov chain the time the process stays in a state is exponentially distributed. The time T in state 0 is thus exponentially distributed with expectation 5 (given in the text) and thus:

$$P(T < 3 | T > 2) \overset{memoryless}{=} P(T < 1) = \int_0^1 (1/5) e^{-t/5} dt = 1 - e^{-1/5} = \underline{\underline{0.18}}$$

b) An overview of the possible transitions is given on the figure below:



By balancing the rate in and out of each state we get:

$$\begin{aligned}
0: &\quad \pi_1 \nu_1 p_{10} + \pi_2 \nu_2 p_{20} = \pi_0 \nu_0 \\
1: &\quad \pi_0 \nu_0 p_{01} = \pi_1 \nu_1 \\
2: &\quad \pi_0 \nu_0 p_{02} + \pi_1 \nu_1 p_{12} = \pi_2 \nu_2
\end{aligned}$$

Inserting $\nu_0 = 1/5 = 0.2$, $\nu_1 = 1/0.25 = 4$, $\nu_2 = 1/0.5 = 2$ and the probabilities given in the text this becomes:

$$\begin{aligned}
0: &\quad 3\pi_1 + 2\pi_2 = 0.2\pi_0 \\
1: &\quad 0.1\pi_0 = 4\pi_1 \\
2: &\quad 0.1\pi_0 + \pi_1 = 2\pi_2
\end{aligned}$$

Solving two of these equations together with $\pi_0 + \pi_1 + \pi_2 = 1$ gives

$$\pi_0 = \frac{80}{\underline{\underline{87}}}, \quad \pi_1 = \frac{2}{\underline{\underline{87}}}, \quad \pi_2 = \frac{5}{\underline{\underline{87}}}$$