Solution exercise set 12

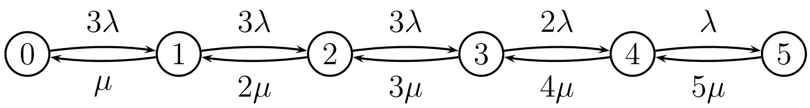
Note on Markov processes, Exercise 14

Since all failure times and all repair times are exponential and independent, the time the process stays in a state (specific number of machines working) will for all states be exponential. Thus, due to the memoryless property of the exponential distribution the process will be a continuous time Markov chain. Moreover, since the repair and failure times are independent *continuous* random variables, two or more repairs or failures can not happen at the same time point. I.e. the number of working machines will only move up or down in steps of one and we thus have a birth and death process. The state space will be $\mathcal{S} = \{0, 1, 2, \ldots, m\}$.

With i working machines, the total rate of failure will be $i\mu$. If ≤ 3 machines are out of order all failed machines are under repair and the total rate of repair is $(m-i)\lambda$ for $i > m-3$. If more than 3 machines are out of order only 3 of them are being repaired and the total repair rate is 3λ. Thus the birth and death rates are:

$$\begin{aligned}
\lambda_i &= \begin{cases} 3\lambda, & i = 0, 1, 2, \ldots, m-3-1, m-3 \\ (m-i)\lambda, & i = m-3+1, m-3+2, \ldots, m \end{cases} \\
\mu_i &= i\mu, \qquad i = 0, 1, 2, \ldots, m
\end{aligned}$$

For $m = 5$, the transition graph is something like



Note on Markov processes, Exercise 15

Define $X(t)$ as the number of components working at time t. With repair and failure times being exponentially distributed the process $\{X(t) : \ t \geq 0\}$ will be a continuous time Markov chain with state space $\mathcal{S} = \{0, 1, 2\}$. The system works then the system is in state 1 or 2. To find the long-run probability the system works we need to find the steady state probabilities.

By balancing the rates in and out of each of the states we get the following:

$$\begin{aligned}
0 &: \quad \pi_1\lambda = \pi_0\mu \\
1 &: \quad \pi_0\mu + \pi_2 2\lambda = \pi_1\mu + \pi_1\lambda \\
2 &: \quad \pi_1\mu = \pi_2 2\lambda
\end{aligned}$$

From these equations we get $\pi_1 = \frac{\mu}{\lambda}\pi_0$ and $\pi_2 = \frac{\mu^2}{2\lambda^2}\pi_0$. Using $\pi_0 + \pi_1 + \pi_2 = 1$ gives

$$\pi_0 = \frac{2\lambda^2}{2\lambda^2 + 2\lambda\mu + \mu^2}$$

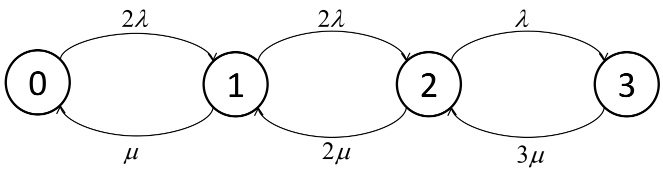
and the probability that the system works is thus:

$$\pi_1 + \pi_2 = 1 - \pi_0 = \underline{\underline{\frac{2\lambda\mu + \mu^2}{2\lambda^2 + 2\lambda\mu + \mu^2}}}$$

Exercise 1:

a) Let $\gamma = 1/8$ be the failure rate and $\lambda = 1/2$ be the repair rate. Since only two machines can be repaired at the same time the total repair rate (or birth rate) in states 0 and 1 are 2λ. Since all working machines can fail, the total failure rate (or death rate) in state i is $i\gamma$. Since the failure and repair times are continuous two failures or repairs can not happen at the same time, and we thus only move on step up or down in the number of working machines. I.e. the process is a birth and death process.

A plot of possible direct transitions with corresponding rates are given below.



Balancing the rate out and the rate in in each state give the following steady state equations:

$$\begin{aligned}
0:&\quad \pi_1\gamma = \pi_0 2\lambda \\
1:&\quad \pi_0 2\lambda + \pi_2 2\gamma = \pi_1\gamma + \pi_1 2\lambda \\
2:&\quad \pi_1 2\lambda + \pi_3 3\gamma = \pi_2 2\gamma + \pi_2\lambda \\
3:&\quad \pi_2\lambda = \pi_3 3\gamma
\end{aligned}$$

Inserting $\gamma = 1/8$ and $\lambda = 1/2$ and using three of these equations we get $\pi_1 = 8\pi_0$, $\pi_2 = 4\pi_1 = 32\pi_0$, $\pi_3 = (4/3)\pi_2 = (128/3)\pi_0$ which inserted in $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$ gives

$$\pi_0 = \underline{\underline{\frac{3}{251}}}, \quad \pi_1 = \underline{\underline{\frac{24}{251}}}, \quad \pi_2 = \underline{\underline{\frac{96}{251}}}, \quad \pi_3 = \underline{\underline{\frac{128}{251}}}.$$

b) Let M be the number of machines in use and let N be the number of specialists working on the machines (and remember that there are only two specialists). Then

$$\begin{aligned} \mathrm{E}(M) &= \sum_m mP(M=m) = 0\cdot\frac{3}{251}+1\cdot\frac{24}{251}+2\cdot\frac{96}{251}+3\cdot\frac{128}{251} = \frac{600}{251} = \underline{\underline{2.39}} \\ \mathrm{E}(N) &= \sum_n nP(N=n) = 0\cdot\frac{128}{251}+1\cdot\frac{96}{251}+2\cdot(\frac{24}{251}+\frac{3}{251}) = \frac{150}{251} = \underline{\underline{0.60}} \end{aligned}$$

The probability that at least one machine is up is $1-\pi_0 = \underline{\underline{\frac{248}{251} \approx 0.988}}$.

Exercise 2:

a) Let $W_5 = Z_1 + Z_2 + \cdots + Z_5$ denote the waiting time until you start to get served. Here Z_i denotes the time from when customer number i in the queue becomes first in the queue until he is being served. (Except Z_1 which is the time from when you enter until the first customer in queue is being served, but due to the memoryless property of the exponential distribution Z_1 is having the same distribution as the other Z_i.)

Alternatively you can think of the Z_is as times between each time a customer is finished - and 5 customers have to finish before it is your time to be served.

Further $Z_i = \min\{V_1, V_2\}$ where V_j is the time until server j is ready to take a new customer. Due to the memoryless property of the exponential distribution, from when customer i moves up to be first in line V_1, V_2 are iid exponential with $\mathrm{E}(V_j) = 4$. We then know that since $Z_i = \min\{V_1, V_2\}$ we have that Z_i is exponentially distributed with $\mathrm{E}(Z_i) = \mathrm{E}(V_j)/2 = 2$. Then:

$$\mathrm{E}(W_5) = \mathrm{E}(Z_1) + \mathrm{E}(Z_2) + \cdots + \mathrm{E}(Z_5) = 5\cdot 2 = \underline{\underline{10}}$$

Let C denote the service time. The time until you are finished is then $W_5 + C$ and

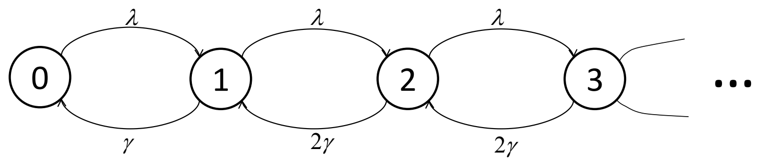
$$\mathrm{E}(W_5 + C) = \mathrm{E}(W_5) + \mathrm{E}(C) = 10 + 4 = \underline{\underline{14}}$$

b) From point a) we have that $W_5 = Z_1 + Z_2 + \cdots + Z_5$ is a sum of 5 independent exponentially distributed variables with mean 2. Then W_5 is gamma distributed with parameters $\alpha = 5$ and $\beta = 2$. We can integrate over this gamma density to find $P(W_5 > 10)$. However, this is a somewhat tricky integral to solve, and an alternative way to solve the problem is the following:

If we look at when customers are finished, we have explained in point a) that the times between when customers are finished is exponentially distributed with expectation 2. This means that the process where we record when customers are finished is a Poisson process with intensity/rate $\lambda = 1/2 = 0.5$. If we let Y be the number of customers who are finished during 10 minutes we have that this number is Poisson distributed with parameter $\lambda t = 0.5\cdot 10 = 5$. If it takes more than 10 minutes before you (customer number 5) start getting served this means that less than 5 customers are finished during the 10 minutes. I.e.:

$$P(W_5 > 10) = P(Y < 5) = P(Y \leq 4) \stackrel{table}{=} \underline{\underline{0.44}}.$$

c) We have a birth and death process with birth rate $\lambda = 0.4$ in all states and death rate of $2\gamma = 0.5$ in all states greater than 1 (since two persons are being served). In state 1 the death rate is $\gamma = 0.25$ (since only one customer is being served) and in state 0 the death rate is of course 0. An overview is given in the figure on the next page.



Balancing the rate out and the rate in in each state give the following steady state equations:

$$\begin{aligned}
0:& \quad \pi_1\gamma = \pi_0\lambda \\
1:& \quad \pi_0\lambda + \pi_2 2\gamma = \pi_1(\lambda+\gamma) \\
2:& \quad \pi_1\lambda + \pi_3 2\gamma = \pi_2(\lambda+2\gamma) \\
3:& \quad \pi_2\lambda + \pi_4 2\gamma = \pi_3(\lambda+2\gamma) \\
\vdots& \qquad\qquad \vdots
\end{aligned}$$

From the first equation we get $\pi_1 = \frac{\lambda}{\gamma}\pi_0$. Combining the two first equations gives $\pi_2 2\gamma = \pi_1\lambda$ which again give $\pi_2 = \frac{\lambda}{2\gamma}\pi_1 = \frac{\lambda^2}{2\gamma^2}\pi_0$. Inserting $\pi_1\lambda = \pi_2 2\gamma$ in the third equation give $\pi_3 = \frac{\lambda}{2\gamma}\pi_2 = \frac{\lambda^3}{2^2\gamma^3}\pi_0$. This continues with the same structure and we generally have:

$$\pi_k = 2\left(\frac{\lambda}{2\gamma}\right)^k \pi_0, \qquad k = 1, 2, 3, \ldots$$

Combining this with $\sum_{k=0}^{\infty} \pi_k = 1$ give:

$$\begin{aligned}
\pi_0 + \sum_{k=1}^{\infty} 2\left(\frac{\lambda}{2\gamma}\right)^k \pi_0 &= 1 \\
\Rightarrow \quad \pi_0 &= \frac{1}{1 + 2\sum_{k=1}^{\infty}\left(\frac{\lambda}{2\gamma}\right)^k} = \frac{1}{1 + 2\frac{\lambda/(2\gamma)}{1-\lambda/(2\gamma)}} = \frac{2\gamma-\lambda}{2\gamma+\lambda} \\
\pi_k &= 2\left(\frac{\lambda}{2\gamma}\right)^k \frac{2\gamma-\lambda}{2\gamma+\lambda}
\end{aligned}$$

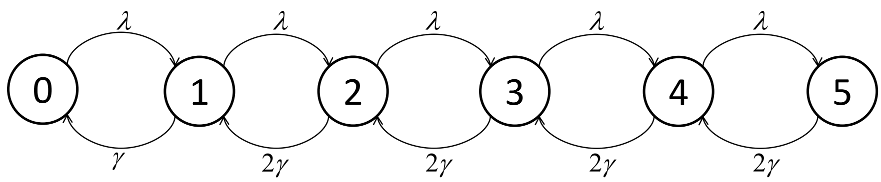
Inserting $\lambda = 0.4$ and $\gamma = 0.25$ we get

$$\pi_0 = \underline{\underline{\frac{1}{9}}}, \quad \pi_k = \underline{\underline{\frac{2}{9}\left(\frac{4}{5}\right)^k}} \qquad k = 1, 2, 3, \ldots$$

Let K denote the number of customers. The expected number of customers in steady state:

$$\mathrm{E}(K) = \sum_{k=0}^{\infty} kP(K=k) = \sum_{k=0}^{\infty} k\pi_k = \sum_{k=1}^{\infty} k\frac{2}{9}\left(\frac{4}{5}\right)^k = \frac{2}{9}\sum_{k=1}^{\infty} k\left(\frac{4}{5}\right)^k = \frac{2}{9}\frac{4/5}{(1-4/5)^2} = \underline{\underline{\frac{40}{9}}}$$

d) We have a birth and death process with state space $\{0,1,2,3,4,5\}$. The birth rate is $\lambda = 0.4$ in all states except state 5 which have birth rate 0. The death rate is $2\gamma = 0.5$ in all states greater than 1 (since two persons are being served). In state 1 the death rate is $\gamma = 0.25$ (since only one customer is being served) and in state 0 the death rate is of course 0. An overview is given in the figure below.



Balancing the rate out and the rate in in each state give the following steady state equations:

$$\begin{aligned}
0:&\quad \pi_1\gamma = \pi_0\lambda \\
1:&\quad \pi_0\lambda + \pi_2 2\gamma = \pi_1(\lambda+\gamma) \\
2:&\quad \pi_1\lambda + \pi_3 2\gamma = \pi_2(\lambda+2\gamma) \\
3:&\quad \pi_2\lambda + \pi_4 2\gamma = \pi_3(\lambda+2\gamma) \\
4:&\quad \pi_3\lambda + \pi_5 2\gamma = \pi_4(\lambda+2\gamma) \\
5:&\quad \pi_4\lambda = \pi_5 2\gamma
\end{aligned}$$

From the first equation we get $\pi_1 = \frac{\lambda}{\gamma}\pi_0$. Combining the two first equations gives $\pi_2 2\gamma = \pi_1\lambda$ which again give $\pi_2 = \frac{\lambda}{2\gamma}\pi_1 = \frac{\lambda^2}{2\gamma^2}\pi_0$. Inserting $\pi_1\lambda = \pi_2 2\gamma$ in the third equation gives $\pi_3 = \frac{\lambda}{2\gamma}\pi_2 = \frac{\lambda^3}{2^2\gamma^3}\pi_0$. This continues with the same structure and we have:

$$\pi_k = 2\left(\frac{\lambda}{2\gamma}\right)^k \pi_0, \qquad k = 1,2,3,4,5$$

Combining this with $\sum_{k=0}^5 \pi_k = \pi_0 + \sum_{k=1}^5 2\left(\frac{\lambda}{2\gamma}\right)^k \pi_0 = 1$ gives:

$$\begin{aligned}
\pi_0 &= \frac{1}{1+2\sum_{k=1}^5\left(\frac{\lambda}{2\gamma}\right)^k} = \frac{1}{1+2\frac{\lambda}{2\gamma}\frac{1-(\lambda/(2\gamma))^5}{1-\lambda/(2\gamma)}} \\
&= \frac{1-\frac{\lambda}{2\gamma}}{1-\frac{\lambda}{2\gamma}+2\frac{\lambda}{2\gamma}-2(\frac{\lambda}{2\gamma})^6} = \frac{1-\frac{\lambda}{2\gamma}}{1+\frac{\lambda}{2\gamma}-2(\frac{\lambda}{2\gamma})^6}
\end{aligned}$$

Inserting $\lambda = 0.4$ and $\gamma = 0.25$ we get (notice in partcular that $\lambda/(2\gamma) = 4/5$)

$$\begin{aligned}
\pi_0 &= \frac{1-4/5}{1+4/5-2\cdot(4/5)^6} = 0.157 \approx \underline{\underline{0.16}} \\
\pi_1 &= 2\cdot(4/5)^1\cdot 0.157 = \underline{\underline{0.25}} \\
\pi_2 &= 2\cdot(4/5)^2\cdot 0.157 = \underline{\underline{0.20}} \\
\pi_3 &= 2\cdot(4/5)^3\cdot 0.157 = \underline{\underline{0.16}} \\
\pi_4 &= 2\cdot(4/5)^4\cdot 0.157 = \underline{\underline{0.13}} \\
\pi_5 &= 2\cdot(4/5)^5\cdot 0.157 = \underline{\underline{0.10}}
\end{aligned}$$

The proportion of time customers are asked to call back is $\pi_5 = \underline{\underline{0.10}}$.

The expected number of customers:

$$\mathrm{E}(K) = \sum_{k=0}^{5} kP(K=k) = \sum_{k=0}^{5} k\pi_k = 1\cdot 0.25 + 2\cdot 0.20 + 3\cdot 0.16 + 4\cdot 0.13 + 5\cdot 0.10 = \underline{\underline{2.15}}$$

Exercise 3:

a)

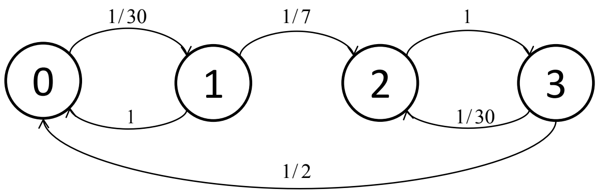
$$\begin{aligned}
P(T_i > 35) &= \int_{35}^{\infty}(1/30)e^{-t/30}dt = e^{-35/30} = \underline{\underline{0.311}} \\
P(T_i > 35|T_i > 25) &\overset{memoryless}{=} P(T_i > 10) = e^{-10/30} = \underline{\underline{0.717}}
\end{aligned}$$

Since $N(t)$ is Poisson distributed with parameter λt we get that $N(30)$ will be Poisson distributed with $\lambda t = (1/30)\cdot 30 = 1$ (that the intensity/rate is $\lambda = 1/30$ follows from the information that the expected time between events is 30) and thus

$$P(N(30) = 2) = \frac{1^2}{2!}e^{-1} = \frac{1^2}{2!}e^{-1} = e^{-1}/2 = \underline{\underline{0.184}}$$

Finally, $E(S_{10}) = \mathrm{E}(T_1) + \ldots \mathrm{E}(T_{10}) = 10\cdot 30 = \underline{\underline{300}}$.

b) The transition graph is displayed in the plot below. In state 0 the only thing that can happen is that A fails and the system goes to state 1. In state 1, either also B fails and the system goes to state 2, or A gets repaired and the system goes to state 0. In state 2 the only thing that can happen is that A gets repaired and the system goes to state 3. In state 3, either also A fails and the system goes to state 2, or B gets repaired and the system goes to state 0. The rates for the various transitions are given in the plot. The



steady state equations become:

$$
\begin{aligned}
0: &\qquad \frac{1}{30}\pi_0 = \pi_1 + \frac{1}{2}\pi_3 \\
1: &\qquad (1+\frac{1}{7})\pi_1 = \frac{1}{30}\pi_0 \\
2: &\qquad \pi_2 = \frac{1}{7}\pi_1 + \frac{1}{30}\pi_3 \\
3: &\qquad (\frac{1}{30}+\frac{1}{2})\pi_3 = \pi_2
\end{aligned}
$$

Using three of these equations and $\pi_0+\pi_1+\pi_2+\pi_3 = 1$ give the solution $\pi_0 = \underline{\underline{\frac{3600}{3751} = 0.96}}$, $\pi_1 = \underline{\underline{\frac{105}{3751} = 0.028}}$, $\pi_2 = \underline{\underline{\frac{16}{3751} = 0.004}}$ and $\pi_3 = \underline{\underline{\frac{30}{3751} = 0.008}}$.

c) The system is not working in state 2, and this happens in $\underline{\underline{16/3751 = 0.004}}$ parts of the time.

The exepcted number of components under repair is:

$$0 \cdot \pi_0 + 1 \cdot (\pi_1 + \pi_3) + 2 \cdot \pi_2 = (105+30)/3751 + 2 \cdot 16/3751 = \underline{\underline{167/3751 = 0.045}}$$

The proportion of time A is under repair is

$$\pi_1 + \pi_2 = (105+16)/3751 = \underline{\underline{121/3751 = 0.032}}$$