STA500 Introduction to Probability and Statistics 2, autumn 2018.

Solution exercise set 1

Exercises from the book:

3.42

The conditional pdf of X given Y is: $f(x|y) = f(x,y)/h(y)$, i.e. we have to find $h(y)$:

$$h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{0}^{\infty} e^{-(x+y)}dx = e^{-y}\int_{0}^{\infty} e^{-x}dx = e^{-y}[-e^{-x}]_0^{\infty} = e^{-y}$$

I.e. we get that $f(x|y) = e^{-(x+y)}/e^{-y} = e^{-x}$ (and from this we see that X and Y are independent since $f(x|y)$ does not depend on y).

$$P(0 < X < 1|Y = 2) = \int_0^1 f(x|y)dx = \int_0^1 e^{-x}dx = [-e^{-x}]_0^1 = -e^{-1} + 1 = \underline{\underline{0.632}}.$$

3.68/3.70

Joint pdf:

$$f(x,y) = \begin{cases} \frac{3x-y}{9} & \text{,for } 1 < x < 3,\ 1 < y < 2 \\ 0 & \text{,otherwise} \end{cases}$$

a) The (marginal) distributions of X and Y:

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy = \int_1^2 f(x,y)\,dy = \underline{\underline{\frac{2x-1}{6}}} \ ,\quad 1 < x < 3 \\ h(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx = \int_1^3 f(x,y)\,dx = \underline{\underline{\frac{12-2y}{9}}} \ ,\quad 1 < y < 2 \end{aligned}$$

b) Are X and Y independent ?

$$g(x)\cdot h(y) = \frac{(2x-1)(6-y)}{27} \neq f(x,y)$$

I.e. X and Y are <u>not independent</u> (in other words they are dependent).

c)

$$P(X > 2) = \int_2^{\infty} g(x)\,dx = \int_2^3 g(x)\,dx = \int_2^3 \frac{2x-1}{6}\,dx = \underline{\underline{\frac{2}{3}}}$$

4.55/4.53

Let W=profit. From the information given in the exercise we get that
$W = 1.65\cdot X - 5\cdot 1.20 + 0.75\cdot 1.20\cdot(5-X) = 0.75\cdot X - 1.5$.

$$\mathrm{E}(X) = \sum_x xf(x) = 0\cdot\frac{1}{15} + 1\cdot\frac{2}{15} + 2\cdot\frac{2}{15} + 3\cdot\frac{3}{15} + 4\cdot\frac{4}{15} + 5\cdot\frac{3}{15} = \frac{46}{15}$$

Hence: $\mathrm{E}(W) = 0.75\mathrm{E}(X) - 1.5 = 0.75\cdot\frac{46}{15} - 1.5 = \underline{\underline{0.8}}$

4.62/4.64

$Z = -2X + 4Y - 3$, due to independence $\text{Cov}(X,Y) = 0$ and we get:

$$\text{Var}(Z) = (-2)^2\text{Var}(X) + 4^2\text{Var}(Y) = 4 \cdot 5 + 16 \cdot 3 = \underline{\underline{68}}$$

4.63/4.65

When the variables are dependent we must take into account the covariance:

$$\text{Var}(Z) = (-2)^2\text{Var}(X) + 4^2\text{Var}(Y) + 2 \cdot (-2) \cdot 4\text{Cov}(X,Y) = 4 \cdot 5 + 16 \cdot 3 - 16 \cdot 1 = \underline{\underline{52}}$$

4.90/4.92

$$\rho_{XY} = \text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}} = \frac{1}{\sqrt{5}\sqrt{3}} = \underline{\underline{0.258}}$$

Exercise 1:

$$f(x) = \begin{cases} 4x(1-x^2) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a)

$$P(X < 0.5) = \int_0^{0.5} 4(x - x^3)dx = 4[\frac{1}{2}x^2 - \frac{1}{4}x^4]_0^{0.5} = 4(\frac{1}{2}0.5^2 - \frac{1}{4}0.5^4) = \underline{\underline{\frac{7}{16} = 0.4375}}$$

$$\begin{aligned}
\text{E}(X) &= \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x4(x - x^3)dx = 4\int_0^1 (x^2 - x^4)dx \\
&= 4[\frac{1}{3}x^3 - \frac{1}{5}x^5]_0^1 = 4[\frac{1}{3} - \frac{1}{5}] = 4 \cdot \frac{2}{15} = \underline{\underline{\frac{8}{15} = 0.533}} \\
\text{Var}(X) &= E(X^2) - [E(X)]^2 \\
\text{E}(X^2) = &= \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2 4(x - x^3)dx = 4\int_0^1 (x^3 - x^5)dx \\
&= 4[\frac{1}{4}x^4 - \frac{1}{6}x^6]_0^1 = 4[\frac{1}{4} - \frac{1}{6}] = 4 \cdot \frac{1}{12} = \frac{1}{3} \\
\text{Var}(X) &= \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \underline{\underline{0.049}}
\end{aligned}$$

b)

$$\begin{aligned}
F(x) &= P(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_0^x 4t(1 - t^2)dt \\
&= \int_0^x (4t1 - 4t^3)dt = [2t^2 - t^4]_0^x = \underline{\underline{2x^2 - x^4}} \\
P(X < 0.3) &= F(0.3) = 2 \cdot 0.3^2 - 0.3^4 = \underline{\underline{0.1719}}
\end{aligned}$$

Exercise 2:

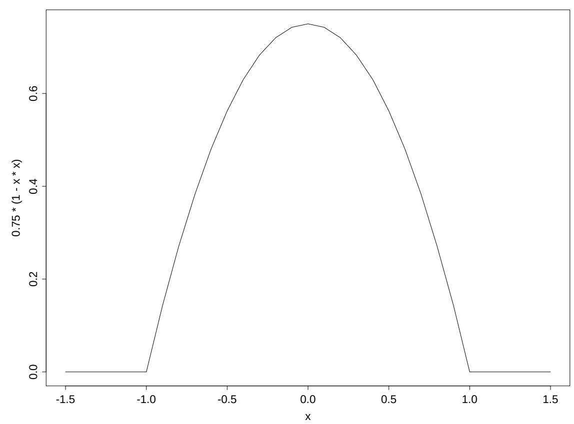
$$f(x) = \begin{cases} k(1-x^2) & \text{for } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) For $f(x)$ to be a pdf we must have $\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{1} f(x)dx = 1$.

$$\int_{-1}^{1} f(x)dx = k[x - \frac{1}{3}x^3]_{-1}^{1} = k(1 - \frac{1}{3} - (-1 + \frac{1}{3})) = k\frac{4}{3} = 1$$

This gives $\underline{\underline{k = \frac{3}{4}}}$.

b) A sketch of $f(x)$ is given in the figure below



c)

$$P(X \le 0.5) = \int_{-1}^{0.5} \frac{3}{4}(1-x^2)dx = \frac{3}{4}[x - \frac{1}{3}x^3]_{-1}^{0.5} = \frac{3}{4}(0.5 - 0.04167 - (-1 + 0.3333)) = \underline{\underline{0.8438}}$$

$$P(X \le 0.8 | X > 0.5) = \frac{P(X \le 0.8 \cap X > 0.5)}{P(X > 0.5)} = \frac{P(0.5 < X \le 0.8)}{P(X > 0.5)}$$

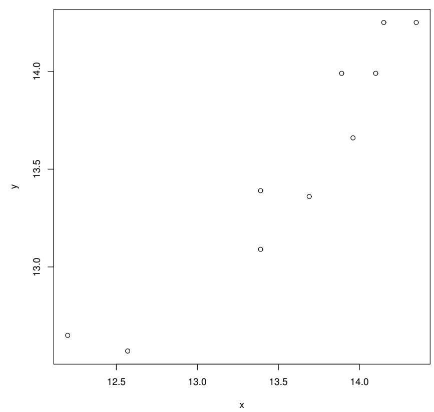
$$P(0.5 < X \le 0.8) = \int_{0.5}^{0.8} \frac{3}{4}(1 - x^2)dx = \frac{3}{4}[x - \frac{1}{3}x^3]_{0.5}^{0.8} = \frac{3}{4}(0.629 - 0.458) = 0.128$$

This gives $P(X \le 0.8 | X > 0.5) = \frac{0.128}{1-0.8438} = \underline{\underline{0.821}}$

$$\begin{aligned} \mathrm{E}(Y) &= E(1 + X^2) = \int_{-\infty}^{\infty} (1 + x^2) f(x)dx = \int_{-1}^{1} (1 + x^2)\frac{3}{4}(1 - x^2)dx \\ &= \frac{3}{4}\int_{-1}^{1} (1 - x^4)dx = \frac{3}{4}[x - \frac{1}{5}x^5]_{-1}^{1} = \frac{3}{4}[2 - \frac{2}{5}] = \underline{\underline{1.2}} \end{aligned}$$

Exercise 3:

a)



b)

$$\begin{aligned} s_x^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{1}{10-1}4.436 = 0.493 \\ s_y^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(y_i-\bar{y})^2 = \frac{1}{10-1}3.414 = 0.379 \\ s_{xy} &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}) = \frac{1}{10-1}3.670 = 0.408 \\ r_{xy} &= \frac{s_{xy}}{\sqrt{s_x^2 \cdot s_y^2}} = \frac{0.408}{\sqrt{0.493\cdot 0.379}} = \underline{\underline{0.94}} \end{aligned}$$

A strong positive correlation - the gas prices follow each other to a large degree.

Exercise 4:

$$\begin{aligned} \mathrm{E}(\bar{X}) &= \frac{1}{n}\mathrm{E}(\sum_{i=1}^{n}X_i) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}(X_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n}n\mu = \underline{\underline{\mu}} \\ \mathrm{Var}(\bar{X}) &= \left(\frac{1}{n}\right)^2\mathrm{Var}(\sum_{i=1}^{n}X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n^2}n\sigma^2 = \underline{\underline{\frac{\sigma^2}{n}}} \\ \mathrm{E}(Y) &= \mathrm{E}(\sum_{i=1}^{n}a_iX_i) = \sum_{i=1}^{n}a_i\mathrm{E}(X_i) = \sum_{i=1}^{n}a_i\mu = \underline{\underline{\mu\sum_{i=1}^{n}a_i}} \\ \mathrm{Var}(Y) &= \mathrm{Var}(\sum_{i=1}^{n}a_iX_i) = \sum_{i=1}^{n}a_i^2\mathrm{Var}(X_i) = \sum_{i=1}^{n}a_i^2\sigma^2 = \underline{\underline{\sigma^2\sum_{i=1}^{n}a_i^2}} \\ \mathrm{E}(Z) &= \mathrm{E}\left(\frac{\sum_{i=1}^{n}a_iX_i}{\sum_{i=1}^{n}a_i}\right) = \frac{\sum_{i=1}^{n}a_i\mathrm{E}(X_i)}{\sum_{i=1}^{n}a_i} = \frac{\sum_{i=1}^{n}a_i\mu}{\sum_{i=1}^{n}a_i} = \mu\frac{\sum_{i=1}^{n}a_i}{\sum_{i=1}^{n}a_i} = \underline{\underline{\mu}} \end{aligned}$$

$$
\begin{aligned}
\mathrm{Var}(Z) &= \mathrm{Var}\left(\frac{\sum_{i=1}^n a_i X_i}{\sum_{i=1}^n a_i}\right) = \frac{1}{(\sum_{i=1}^n a_i)^2}\mathrm{Var}(\sum_{i=1}^n a_i X_i) = \frac{1}{(\sum_{i=1}^n a_i)^2}\sum_{i=1}^n a_i^2 \mathrm{Var}(X_i) \\
&= \frac{\sum_{i=1}^n a_i^2 \sigma^2}{(\sum_{i=1}^n a_i)^2} = \underline{\underline{\sigma^2 \frac{\sum_{i=1}^n a_i^2}{(\sum_{i=1}^n a_i)^2}}}
\end{aligned}
$$