STA500 Introduction to Probability and Statistics 2, autumn 2018.

Solution exercise set 2

Exercises from the book:

5.58/5.60

Let X be the number of hurricanes per year, and assume that $X \sim \text{Poisson}\,(6)$.

a)

$$P\,(X < 4) = P\,(X \leq 3) = \sum_{x=0}^{3} \frac{6^x}{x!} e^{-6} \overset{\text{table}}{=} 0.1512$$

Or by direct calculation:

$$P\,(X < 4) = P\,(X \leq 3) = \frac{6^0}{0!}e^{-6} + \frac{6^1}{1!}e^{-6} + \frac{6^2}{2!}e^{-6} + \frac{6^3}{3!}e^{-6} = 0.1512$$

b)

$$\begin{aligned} P\,(6 \leq X \leq 8) &= P\,(X \leq 8) - P\,(X \leq 5) = \sum_{x=0}^{8} \frac{6^x}{x!} e^{-6} - \sum_{x=0}^{5} \frac{6^x}{x!} e^{-6} \\ &\overset{\text{table}}{=} 0.8472 - 0.4457 = 0.4015 \end{aligned}$$

5.61/5.65

Let X="the number of 10000 forms containing an error". We then do $n = 10000$ independent trials where we in each trial are checking if the form is containing an error or not and the probability of selecting a form with an error is the same, $p = 1/1000$, in each trial. I.e. $X \sim B(10000, 0.001)$. Since we are in a situation with a large n and a small p we can use the approximation to the Poisson distribution (makes the calculations simpler). X will be approximately Poisson distributed with expectation $\mu = np = 10000 \cdot 0.001 = 10$

$$P\,(6 \leq X \leq 8) = P\,(X \leq 8) - P\,(X \leq 5) \overset{\text{table}}{=} 0.3328 - 0.0671 = \underline{\underline{0.2657}}$$

5.92/5.96

Let Y="the number of kids until the second boy". Y is then having a negative binomial distribution with $k = 2$ and $p = 0.5$.

$$P(Y = 4) = \binom{4-1}{2-1} \cdot 0.5^2 \cdot 0.5^{4-2} = \underline{\underline{0.1875}}$$

6.15/6.11

$X \sim N(24, 3.8^2)$

a)

$$\begin{aligned} P(X \geq 30) &= 1 - P(X < 30) = 1 - P\left(\tfrac{X-24}{3.8} < \tfrac{30-24}{3.8}\right) = 1 - P(Z < 1.58) \\ &= 1 - 0.9429 = \underline{\underline{0.0571}} \end{aligned}$$

b)

$$\begin{aligned} P(X \geq 15) &= 1 - P(X < 15) = 1 - P\left(\frac{X-24}{3.8} < \frac{15-24}{3.8}\right) \\ &= 1 - P(Z < -2.37) = 1 - 0.0089 = 0.9911 \end{aligned}$$

He will be late $\underline{\underline{99.11\%}}$ of the days.

c)

$$\begin{aligned} P(X \geq 25) &= 1 - P(X < 25) = 1 - P\left(\frac{X-24}{3.8} < \frac{25-24}{3.8}\right) = 1 - P(Z < 0.26) \\ &= 1 - 0.6026 = \underline{\underline{0.3974}} \end{aligned}$$

d) k = the length of time above which we find 15% of the trips.

$P(X \geq k) = 1 - P(X < k) = 1 - P(\frac{x-24}{3.8} < \frac{k-24}{3.8}) = 1 - P(Z < \frac{k-24}{3.8}) = 0.15$

$P(Z < \frac{k-24}{3.8}) = 0.85$

$\frac{k-24}{3.8} = 1.04 \quad \Rightarrow \quad k = 3.8 \cdot 1.04 + 24 = 27.95 = \underline{\underline{\text{27 minutes, 57 seconds.}}}$

e) Y = the number of trips among the next 3 which takes more than 30 minutes.

$p = P(X \geq 30) = 0.057$

$Y \sim B(3, 0.057)$

$P(Y = 2) = \binom{3}{2} 0.057^2 (1 - 0.057)^{3-2} = 3 \cdot 0.057^2 \cdot 0.943 = \underline{\underline{0.0092}}$

Exercise 1:

a) We have:

- A specified number of trials, $n = 12$ patients.
- The trials are independent, the patients catch the infection or not independent of each other.
- We are only checking "success" or not "success" in each trial (if a patient is infected or not).
- The probability of "success" is the same in each trial, $p = 0.25$ (all patients are assumed to have the same probability of catching the infection).

I.e., the conditions for the binomial distribution are fulfilled, and we thus have that
X = "the number of the 12 patients catching the disease" is binomially distributed with parameters $n = 12$ and $p = 0.25$, $X \sim B(12, 0.25)$.

b)

$P(X = x) = \binom{12}{x}(0.25)^x(0.75)^{12-x}$ which gives $P(X = 0) = \binom{12}{0}(0.25)^0(0.75)^{12} = \underline{\underline{0.032}}$.

Further $P(X = 1) = \binom{12}{1}(0.25)^1(0.75)^{11} = 0.127$ and hence:

$$P(X > 1) = 1 - P(X \leq 1) = 1 - [P(X = 0) + P(X = 1)] = 1 - 0.032 - 0.127 = \underline{\underline{0.841}}.$$

c)

$\mathrm{E}(X) = np = 12 \cdot 0.25 = \underline{\underline{3}}$, and $\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{np(1-p)} = \sqrt{12 \cdot 0.25 \cdot 0.75} = \underline{\underline{1.5}}$.

Exercise 2:

a) With $\mu = \lambda t = 2 \cdot 1$ we get $P(X = x) = \frac{2^x}{x!} e^{-2}$ which gives $P(X = 2) = \frac{2^2}{2!} e^{-2} = \underline{\underline{0.271}}$. Further $P(X = 0) = \frac{2^0}{0!} e^{-2} = 0.135$ and $P(X = 1) = \frac{2^1}{1!} e^{-2} = 0.271$, and thus

$$\begin{aligned} P(X \geq 3) &= 1 - P(X < 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \\ &= 1 - 0.135 - 0.271 - 0.271 = \underline{\underline{0.323}}. \end{aligned}$$

Alternatively $P(X \leq 2)$ can be found from the table.

b) When we look at a period of 0.5 years we get that the number of accidents is Poisson distributed with $\mu = \lambda t = 2 \cdot 0.5 = 1$, i.e. $P(X = 2) = \frac{1^2}{2!} e^{-1} = \underline{\underline{0.184}}$

c) Over a ten years period the number of accidents is Poisson distributed with $\mu = \lambda t = 2 \cdot 10 = 20$, i.e. $P(X = x) = \frac{20^x}{x!} e^{-20}$. We can use this to calculate $P(X \geq 20) = 1 - P(X \leq 19)$ exactly, but here it is more convenient to use the approximation to the normal distribution. Since $\mu > 15$ the normal approximation is good, and we get

$$\begin{aligned} P(X \geq 20) &= 1 - P(X \leq 19) = 1 - P(\frac{X - 20}{\sqrt{20}} \leq \frac{19 + 0.5 - 20}{\sqrt{20}}) \\ &= 1 - P(Z \leq -0.11) = 1 - 0.4562 = \underline{\underline{0.5438}} \end{aligned}$$

d)

$$P(X > 2 | X \geq 1) = \frac{P(X > 2 \cap X \geq 1)}{P(X \geq 1)} = \frac{P(X > 2)}{1 - P(X = 0)} = \frac{P(X \geq 3)}{1 - 0.135} = \frac{0.323}{0.865} = \underline{\underline{0.373}}.$$

Exercise 3:

$$\begin{aligned} P(X_1 > 800) &= 1 - P(X_1 \leq 800) = 1 - P(Z \leq \frac{800 - 750}{25}) \\ &= 1 - P(Z \leq 2) = 1 - 0.9772 = \underline{\underline{0.0228}} \\ P(X_1 \leq 800 | X_2 > 750) &\overset{indep.!}{=} P(X_1 \leq 800) = \underline{\underline{0.9772}} \\ P(X_1 + X_2 > 1600) &= 1 - P(X_1 + X_2 \leq 1600) \\ &= 1 - P(\frac{X_1 + X_2 - \mathrm{E}(X_1 + X_2)}{\sqrt{\mathrm{Var}(X_1 + X_2)}} \leq \frac{1600 - \mathrm{E}(X_1 + X_2)}{\sqrt{\mathrm{Var}(X_1 + X_2)}}) \\ &= 1 - P(Z \leq \frac{1600 - 2 \cdot 750}{\sqrt{2 \cdot 25^2}}) = 1 - P(Z \leq 2.83) \\ &= 1 - 0.9977 = \underline{\underline{0.0023}} \end{aligned}$$

Exercise 4:

a)

$$\begin{aligned} P(X < 7.5) &= P(Z < \frac{7.5 - 8.2}{0.4}) = P(Z < -1.75) = \underline{\underline{0.04}} \\ P(X > 8.2 | X > 7.5) &= \frac{P(X > 8.2 \cap X > 7.5)}{P(X > 7.5)}) = \frac{P(X > 8.2)}{1 - 0.04} = \frac{1 - P(Z \leq 0)}{0.04} = \frac{0.5}{0.96} = \underline{\underline{0.52}} \end{aligned}$$

b)

We have a situation with

- Independent trials (independent capacity of the beams).
- We check "success" or not "success" in each trial (can carry 7.5 tons or can not) until we have found 20 "successes".
- The probability of "success" is the same in all trials, $p = P(X > 7.5) = 1 - 0.04 = 0.96$.

Y is then having a $\underline{\underline{\text{negative binomial distribution with } k = 20 \text{ and } p = 0.96}}$.

$$\begin{aligned} P(Y < 22) &= P(Y \leq 21) = P(Y = 20) + P(Y = 21) \\ &= (\genfrac{}{}{0pt}{}{20-1}{20-1})0.96^{20}(1-0.96)^{20-20} + (\genfrac{}{}{0pt}{}{21-1}{20-1})0.96^{20}(1-0.96)^{21-20} \\ &= 0.442 + 0.354 = \underline{\underline{0.796}} \end{aligned}$$

$$\mathrm{E}(X) = \frac{k}{p} = \frac{20}{0.96} = \underline{\underline{20.8}}$$