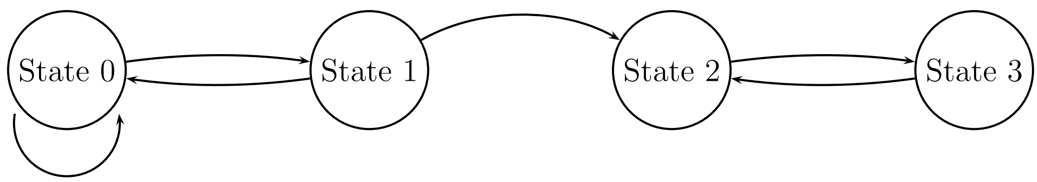
STA500 Introduction to Probability and Statistics 2, autumn 2018.

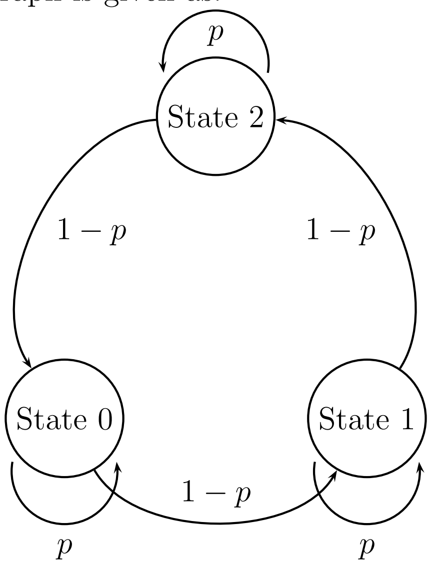
Solution exercise set 9

Note on Markov processes, Exercise 1
We start by drawing the transition graph:



The Markov process has classes $\{0, 1\}$ and $\{2, 3\}$ where the former is transient and the latter is recurrent.

Note on Markov processes, Exercise 2
The process is a Markov chain as the process has a discrete state space $0, 1, 2$, and the distribution of $X_{t+1}|X_t, X_{t-1}, \dots$ depends only on X_t and not on earlier values. The transition graph is given as:

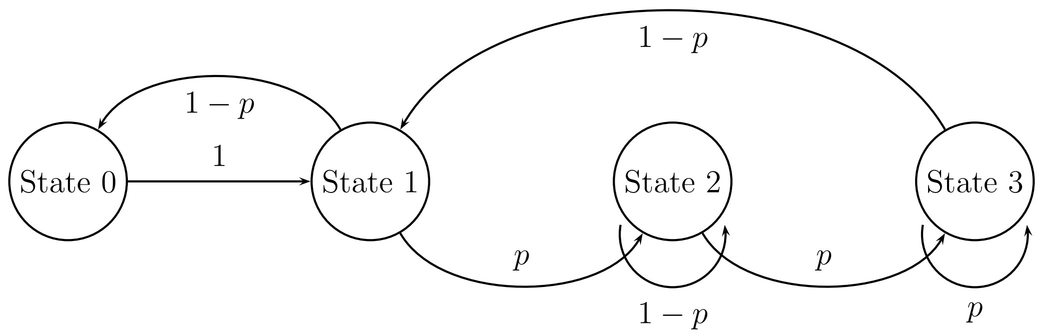


The transition probability matrix is given as:

$$P = \begin{pmatrix} p & 1-p & 0 \\ 0 & p & 1-p \\ 1-p & 0 & p \end{pmatrix}.$$

As p is assumed to be < 1, the chain is irreducible, i.e. one class. As p is assumed to be > 0, the process is aperiodic (e.g. all of the diagonal elements of P are non-zero).

Note on Markov processes, Exercise 3
The transition graph is given as:



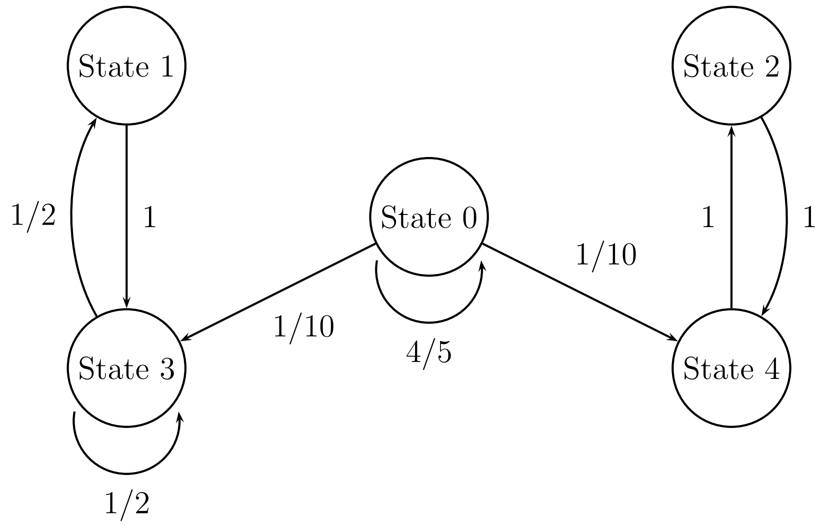
The transition probability matrix is given as

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & p \\ 0 & 1-p & 0 & p \end{pmatrix}$$

The process is irreducible (all states communicate) and aperiodic (e.g. state 3 has a diagonal element in P) when $0 < p < 1$.

Note on Markov processes, Exercise 4

The transition graph is something like



The process has three classes, and is therefore not irreducible.

- Class $\{0\}$ transient.
- Class $\{1,3\}$ recurrent, aperiodic
- Class $\{2,4\}$ recurrent, period 2

Note on Markov processes, Exercise 5
The process is irreducible as both states are accessible from the other one, and state 0 can be revisited in each of $1, 2, 3, 4, 5, \ldots$ transitions so the chain is aperiodic. The steady state probabilities are found by writing down the steady state equations

$$\begin{aligned} \pi_0 &= p\pi_0 + \pi_1 \\ \pi_1 &= (1-p)\pi_0 \\ 1 &= \pi_0 + \pi_1 \end{aligned}$$

Taking the latter of the former two and substitute for π_1 in the normalisation equation, we arrive at

$$1 = (1 + 1 - p)\pi_0 \Rightarrow \pi_0 = \frac{1}{2-p},$$

and finally $\pi_1 = (1-p)\pi_0 = \frac{1-p}{2-p}$. We see that p controls how frequently state 1 is visited. For p close to 0, the process spends roughly half the time in each state (and becomes periodic as $p \to 0$). On the other hand, for p close to 1 the process spends most of the time in state 0 (and state 0 becomes absorbing as $p \to 1$).

Exercise 1:

a) From the information given we have that $p_{11} = P(X_{n+1} = 1|X_n = 1) = 0.75$ and thus $p_{10} = 1 - 0.75 = 0.25$. We also have that $p_{01} = P(X_{n+1} = 1|X_n = 0) = 0.35$ and thus $p_{00} = 1 - 0.35 = 0.65$. The transition probability matrix then becomes:

$$\begin{aligned} P &= \underline{\underline{\begin{pmatrix} 0.65 & 0.35 \\ 0.25 & 0.75 \end{pmatrix}}} \\ P^2 &= P \cdot P = \begin{pmatrix} 0.65 \cdot 0.65 + 0.35 \cdot 0.25 & 0.65 \cdot 0.35 + 0.35 \cdot 0.75 \\ 0.25 \cdot 0.65 + 0.75 \cdot 0.25 & 0.25 \cdot 0.35 + 0.75 \cdot 0.75 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0.51 & 0.49 \\ 0.35 & 0.65 \end{pmatrix}}} \end{aligned}$$

b) From the transition probability matrices we directly read:

$$\begin{aligned} p_{10} &= \underline{\underline{0.25}} \\ p_{01}^2 &= \underline{\underline{0.49}} \\ p_{10}^2 &= \underline{\underline{0.35}} \end{aligned}$$

c) Since X_n contains information about the weather on day n and day $n-1$ knowing $X_{n-1}, X_{n-2}, \ldots$ does not add anything to our knowledge regarding the weather on the next day $n+1$ since it is assumed in the text that the weather only depend on the weather the previous two days. Thus the probability of X_{n+1} only depends on X_n and not on $X_{n-1}, X_{n-2}, \ldots$ - i.e. the process is a Markov chain.

Further, from the information given in the text we have:

$$\begin{aligned}
p_{02} &= P(X_{n+1} = 2|X_n = 0) = P(X_{n+1} = sr|X_n = ss) = 0.3 \\
p_{12} &= P(X_{n+1} = 2|X_n = 1) = P(X_{n+1} = sr|X_n = rs) = 0.5 \\
p_{23} &= P(X_{n+1} = 3|X_n = 2) = P(X_{n+1} = rr|X_n = sr) = 0.6 \\
p_{33} &= P(X_{n+1} = 3|X_n = 3) = P(X_{n+1} = rr|X_n = rr) = 0.8
\end{aligned}$$

Since the weather the next day is either sun or rain there are always two possibilities for the next state. The probabilities given above are the probabilities for going to a state with rain on the next day. The probabilities of going to a state with sun is then:

$$\begin{aligned}
p_{00} &= P(X_{n+1} = 0|X_n = 0) = P(X_{n+1} = ss|X_n = ss) = 1 - 0.3 = 0.7 \\
p_{10} &= P(X_{n+1} = 0|X_n = 1) = P(X_{n+1} = ss|X_n = rs) = 1 - 0.5 = 0.5 \\
p_{21} &= P(X_{n+1} = 1|X_n = 2) = P(X_{n+1} = rs|X_n = sr) = 1 - 0.6 = 0.4 \\
p_{31} &= P(X_{n+1} = 1|X_n = 3) = P(X_{n+1} = rs|X_n = rr) = 1 - 0.8 = 0.2
\end{aligned}$$

The transition probability matrix then becomes as given in the exercise text.

d) From the transition probability matrices we directly read:

$$\begin{aligned}
p_{21} &= P(X_{n+1} = rs|X_n = sr) = \underline{\underline{0.4}} \\
p_{03}^2 &= P(X_{n+2} = rr|X_n = ss) = \underline{\underline{0.18}} \\
p_{11}^2 &= P(X_{n+2} = rs|X_n = rs) = \underline{\underline{0.20}}
\end{aligned}$$

Exercise 2:

a) We read directly from the transition probability matrix that $P(X_1 = 2|X_0 = 0) = \underline{\underline{0.1}}$ and $P(X_4 = 1|X_3 = 0, X_2 = 1) = P(X_4 = 1|X_3 = 0) = \underline{\underline{0.2}}$. From the squared transition probability matrix we get the two step probabilities, and read out directly that $P(X_2 = 1|X_0 = 1) = \underline{\underline{0.39}}$ and $P(X_{21} = 1|X_{19} = 2) = \underline{\underline{0.42}}$

b) Steady state equations:

$$\begin{aligned}
0.7\pi_0 + 0.3\pi_1 + 0.1\pi_2 &= \pi_0 \\
0.2\pi_0 + 0.5\pi_1 + 0.4\pi_2 &= \pi_1 \\
\pi_0 + \pi_1 + \pi_2 &= 1
\end{aligned}$$

Solving this set of equations gives the solution $\pi_0 = \underline{\underline{\frac{17}{40}}}$, $\pi_1 = \underline{\underline{\frac{14}{40}}}$, and $\pi_2 = \underline{\underline{\frac{9}{40}}}$. As n gets large the starting point is forgotten and $P(X_n = 0|X_0 = 0) \to \pi_0 = \underline{\underline{\frac{17}{40}}}$.