Solution exam February 10, 2014

Problem 1:

a) The number of jobs during $t = 1$ minute, X, is Poisson-distributed with expectation $\lambda t = 20$.

$$P(X = 25) = \frac{20^{25}}{25!}e^{-20} = \underline{\underline{0.045}}$$

Since the expectation of the Poisson distributed is larger than 15 it can be approximated by the normal distribution.

$$\begin{aligned} P(X > 25) &= 1 - P(X \le 25) \approx 1 - P(Z \le \frac{25 + 0.5 - \mathrm{E}(X)}{\sqrt{\mathrm{Var}(X)}}) \\ &= 1 - P(Z \le \frac{25 + 0.5 - 20}{\sqrt{20}}) = 1 - P(Z \le 1.23) = 1 - 0.8907 = \underline{\underline{0.11}} \end{aligned}$$

The execution time, T, is exponentially distributed with expectation $\beta = 10$. Then we get:

$$P(T < 5) = \int_0^5 \frac{1}{10}e^{-t/10}dt = [-e^{-t/10}]_0^5 = -e^{-5/10} + 1 = \underline{\underline{0.39}}$$

b) The queue is stable as long as the rate of arrivals to the queue is smaller than the maximum rate of departures from the queue. The arrival rate is $\lambda = 20$ per minute. For each processing unit the departure rate is $\gamma = 1/(10/60) = 6$ per minute (per second the rate would be 1/10, but we must have both rates in the same time scale). The maximum departure rate (when all processing units are busy) is then $4 \cdot \gamma = 4 \cdot 6 = 24$ which is larger than the arrival rate, i.e. the queue is stable.

The queue is not stable if $\lambda \ge 4\gamma$, i.e. if $\underline{\underline{\lambda \ge 24}}$.

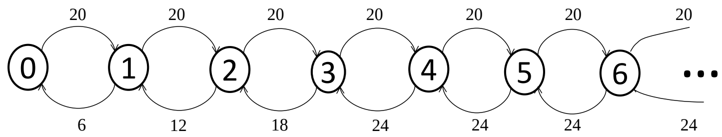
Let $W_6 = Z_1 + Z_2 + \cdots + Z_6$ denote the waiting time until the job starts being executed. Here Z_i denotes the times between each time the queue moves (i.e. a job is finished), and the queue needs to move six times before the job starts being executed. Further due to the memoryless property of the exponential distribution $Z_i = \min\{V_1, V_2, V_3, V_4\}$ where V_1, V_2, V_3, V_4 have the same distribution as the execution time, i.e. they are iid exponential with $\mathrm{E}(V_j) = 10$ and Z_i is thus exponential with $\mathrm{E}(Z_i) = \mathrm{E}(V_j)/4 = 10/4 = 2.5$. Then:

$$\mathrm{E}(W_6) = \mathrm{E}(Z_1) + \mathrm{E}(Z_2) + \cdots + \mathrm{E}(Z_6) = 6 \cdot 2.5 = \underline{\underline{15 \text{ seconds}}} \text{ (or 0.25 minutes)}$$

Let C denote the service time. The time until the job is finished is then $W_6 + C$ and

$$\mathrm{E}(W_6 + C) = \mathrm{E}(W_6) + \mathrm{E}(C) = 15 + 10 = \underline{\underline{25 \text{ seconds}}} \text{ (or 5/12=0.42 minutes)}$$

c) With the arrivals of jobs being according to a Poisson process the time until next arrival is exponentially distributed. Since the exectuion time is exponential and the four processing units work independent the time until next depature is also exponential. Moreover, arrivals and



depatures are independent and the number of jobs in the system will always move up or down in steps of one. I.e., the process is a birth and death process.

The transition graph is displayed in the figure above (if we use seconds as time scale instead of minutes all rates should be divided by 60). Balancing the rate out and the rate in in each state give the following steady state equations for the first five states:

$$\begin{aligned} 0 &: \quad \pi_1 6 = \pi_0 20 \\ 1 &: \quad \pi_0 20 + \pi_2 12 = \pi_1 6 + \pi_1 20 = \pi_1 26 \\ 2 &: \quad \pi_1 20 + \pi_3 18 = \pi_2 12 + \pi_2 20 = \pi_2 32 \\ 3 &: \quad \pi_2 20 + \pi_4 24 = \pi_3 18 + \pi_3 20 = \pi_3 38 \\ 4 &: \quad \pi_3 20 + \pi_5 24 = \pi_4 24 + \pi_4 20 = \pi_4 44 \end{aligned}$$

d) In the long run all units are idle $\pi_o = \underline{\underline{27/1267 = 0.021}}$ of the time.

The long run proportion of time when all four processing units are busy is $\sum_{k=4}^{\infty} \pi_k$, i.e.:

$$\sum_{k=4}^{\infty} \pi_k = 1 - \pi_0 - \pi_1 - \pi_2 - \pi_3 = 1 - 27/1267 - 90/1267 - 150/1267 - 500/3801 = \underline{\underline{2500/3801 = 0.66}}.$$

Let K denote the number of jobs. The expected number of customers in steady state:

$$\begin{aligned} \mathrm{E}(K) &= \sum_{k=0}^{\infty} kP(K = k) = \sum_{k=0}^{\infty} k\pi_k = 0\pi_0 + 1\pi_2 + 2\pi_2 + \sum_{k=3}^{\infty} k\pi_k \\ &= \frac{90}{1267} + 2 \cdot \frac{150}{1267} + \frac{500}{3801} \sum_{k=3}^{\infty} k(\frac{5}{6})^{k-3} \\ &= \frac{390}{1267} + \frac{500}{3801} \sum_{k=3}^{\infty} k(\frac{5}{6})^k(\frac{6}{5})^3 = \frac{390}{1267} + \frac{500}{3801}\frac{6^3}{5^3} \sum_{k=3}^{\infty} k(\frac{5}{6})^k \\ &= \frac{390}{1267} + \frac{4 \cdot 216}{3801}\left[\frac{5/6}{(1 - 5/6)^2} - 5/6 - 2 \cdot 5^2/6^2\right] \\ &= \frac{390}{1267} + \frac{864}{3801}\left[(5/6) \cdot 6^2 - 5/6 - 25/18\right] \\ &= \frac{390}{1267} + \frac{288}{1267}\frac{250}{9} = \frac{390}{1267} + \frac{32 \cdot 250}{1267} = \underline{\underline{\frac{8390}{1267}}} = 6.6. \end{aligned}$$

Problem 2:

a) Since $\text{E}(X) = \text{E}(Y) = 0$ and $\text{Var}(X) = \text{Var}(Y) = \sigma^2$ both X/σ and Y/σ have a $N(0, 1)$ distribution, and thus X^2/σ^2 and Y^2/σ^2 both have a χ^2_1 distribution. Further we have the linear combination results that a sum of independent χ^2-distributed variables is χ^2-distributed with parameter equal to the sum of the parameters for each variable (collection of formulas)- and we thus have that $R^2/\sigma^2 = X^2/\sigma^2 + Y^2/\sigma^2$ has a χ^2_2-distribution.

We have the transformation result that if X has a Weibull distribution with parameters α and β, then $Y = 2\alpha X^\beta$ has a χ^2-distribution with 2 degrees of freedom. If we in our case let $X = R$, $\alpha = 1/(2\sigma^2)$ and $\beta = 2$ then $Y = 2\alpha X^\beta = 2/(2\sigma^2)R^2 = R^2/\sigma^2$, and we have shown above that this quantity has a χ^2_2-distribution. Thus from the transformation result it follows that R must have a Weibull distribution with parameters $\alpha = 1/(2\sigma^2)$ and $\beta = 2$.

If we take the general Weibull density and insert $\alpha = 1/(2\sigma^2)$ and $\beta = 2$ we get

$$f_R(r) = \alpha\beta r^{\beta-1}e^{-\alpha r^\beta} = \frac{1}{2\sigma^2}\cdot 2r^{2-1}e^{-\frac{1}{2\sigma^2}r^2} = \underline{\underline{\frac{r}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}}}}$$

b)

$$\begin{aligned} F_R(r) &= P(R < r) = \int_{-\infty}^{r} f_R(u)du = \int_0^r \frac{r}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}} = [-e^{-\frac{u^2}{2\sigma^2}}]_0^r = \underline{\underline{1 - e^{-\frac{r^2}{2\sigma^2}}}} \\ P(R < 4) &= F_R(4) = 1 - e^{-\frac{4^2}{2\cdot 14^2}} = \underline{\underline{0.040}} \\ P(R > 40) &= 1 - F_R(40) = e^{-\frac{40^2}{2\cdot 14^2}} = \underline{\underline{0.017}} \end{aligned}$$

c) The situation is characterized by

- The outcome of each trial is "success"/not "success" - either she hits the inner zone or not.
- The probability for success is the same in each trial - $p = 0.04$ of hitting the inner zone in each trial.
- Independent trials - independent results of each shot..
- The trials continue until the first success - she continues shooting until the first hit in the inner zone.

W="number of shots until she hits the inner zone" then has a geometric distribution with $p = 0.04$ (special case of the negative binomial distribution where we look at the number of trials until the first success).

$$\begin{aligned} P(W < 5) &= P(W \le 4) = P(W = 1) + P(W = 2) + P(W = 3) + P(W = 4) \\ &= 0.04 + 0.04\cdot(1 - 0.04) + 0.04\cdot(1 - 0.04)^2 + 0.04\cdot(1 - 0.04)^3 = \underline{\underline{0.15}} \\ \text{E}(Y) &= \frac{1}{p} = \frac{1}{0.04} = \underline{\underline{25}} \end{aligned}$$

d)

$$\begin{aligned} F_V(v) &= P(V \le v) = P((R_1 \le v)\cap\cdots\cap(R_m \le v)) \overset{indep.}{=} P(R_1 \le v)\cdots P(R_m \le v) \\ &= F_R(v)^m = \underline{\underline{\left(1 - e^{-\frac{v^2}{2\sigma^2}}\right)^m}} \\ P(V > 40) &= 1 - P(V \le 40) = 1 - \left(1 - e^{-\frac{40^2}{2\cdot 14^2}}\right)^{25} = \underline{\underline{0.347}} \end{aligned}$$

It will be performed 25 independent shots which each either hits or misses the target with the same probability ($p = 0.017$ calculated in b)). The number of these 25 shots which goes outside the target will thus have a binomial distribution with parameters $n = 25$ and $p = 0.017$, and we can use this to calculate the probability that at least one shot goes outside. The probability that at least one shot goes outside is the same as the probability that the worst shot goes outside.

e) MLE for σ:

$$\begin{aligned} L(\sigma) &= \prod_{i=1}^{n}\frac{r_i}{\sigma^2}e^{-\frac{r_i^2}{2\sigma^2}} \\ l(\sigma) &= \ln L(\sigma) = \sum_{i=1}^{n}\ln\left(\frac{r_i}{\sigma^2}e^{-\frac{r_i^2}{2\sigma^2}}\right) = \sum_{i=1}^{n}\left(\ln(r_i) - \ln(\sigma^2) - \frac{r_i^2}{2\sigma^2}\right) \\ &= \sum_{i=1}^{n}\ln(r_i) - 2n\ln(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}r_i^2 \\ \frac{\partial l(\sigma)}{\partial\sigma} &= -\frac{2n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}r_i^2 = 0 \\ & 2n\sigma^2 = \sum_{i=1}^{n}r_i^2 \quad\Rightarrow\quad \sigma^2 = \frac{1}{2n}\sum_{i=1}^{n}r_i^2 \end{aligned}$$

I.e the MLE becomes $\hat{\sigma} = \underline{\underline{\sqrt{\frac{1}{2n}\sum_{i=1}^{n}R_i^2}}}$.

Estimate: $\hat{\sigma} = \sqrt{\frac{1}{2\cdot 50}16017} = \underline{\underline{12.7}}$

f) Since we have as much as $n = 50$ measurements we can calculate a Wald confidence interval, and given the calculations we have already done in e) this will be the easiest appraoch here. We then first find the second derivative of the log-likelihood of σ:

$$\begin{aligned} \frac{\partial l(\sigma)}{\partial\sigma} &= -\frac{2n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}r_i^2 \\ J(\sigma) &= \frac{\partial^2 l(\sigma)}{\partial\sigma^2} = \frac{2n}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{n}r_i^2 \\ J(\hat{\sigma}) &= \frac{2n}{\hat{\sigma}^2} - \frac{3}{\hat{\sigma}^4}2n\frac{1}{2n}\sum_{i=1}^{n}r_i^2 = \frac{2n}{\hat{\sigma}^2} - \frac{6n}{\hat{\sigma}^2} = -\frac{4n}{\hat{\sigma}^2} \end{aligned}$$

The Wald confidence interval then becomes (since $z_{\alpha/2} = z_{0.05} = 1.645$:

$$\begin{aligned} [\hat{\sigma} - z_{\alpha/2}\sqrt{-1/J(\hat{\sigma})}, \hat{\sigma} + z_{\alpha/2}\sqrt{-1/J(\hat{\sigma})}] &= [\hat{\sigma} - z_{\alpha/2}\sqrt{\hat{\sigma}^2/(4n)}, \hat{\sigma} + z_{\alpha/2}\sqrt{\hat{\sigma}^2/(4n)}] \\ &= [12.7 - 1.645\sqrt{12.7^2/(4\cdot 50)}, [12.7 + 1.645\sqrt{12.7^2/(4\cdot 50)}] \\ &= \underline{\underline{[11.2,\ 14.2]}} \end{aligned}$$

Alternatively we can find an exact confidence interval by using the relation to the χ^2-distribution and results for sums of χ^2-distributions.

We see that $2n\hat{\sigma}^2/\sigma^2 = \sum_{i=1}^n R_i^2/\sigma^2$. From point a) we have that R^2/σ^2 has a χ_2^2-distribution. Further we have the result that a sum of independent χ^2-distributed variables is χ^2-distributed with parameter equal to the sum of the parameters for each variable (collection of formulas). I.e. since $\sum_{i=1}^n R_i^2/\sigma^2$ is a sum of n indep. χ_2^2-distributed this quantity will have a χ_{2n}^2-distribution. Further

$$\sum_{i=1}^n R_i^2/\sigma^2 = 2n\frac{1}{2n}\sum_{i=1}^n R_i^2/\sigma^2 = \frac{2n\hat{\sigma}^2}{\sigma^2}$$

Then we get:

$$\begin{aligned}
P(\chi^2_{1-\alpha/2,2n} \leq \frac{2n\hat{\sigma}^2}{\sigma^2} \leq \chi^2_{\alpha/2,2n}) &= 1-\alpha \\
P(\frac{\chi^2_{1-\alpha/2,2n}}{2n\hat{\sigma}^2} \leq \frac{1}{\sigma^2} \leq \frac{\chi^2_{\alpha/2,2n}}{2n\hat{\sigma}^2}) &= 1-\alpha \\
P(\frac{2n\hat{\sigma}^2}{\chi^2_{\alpha/2,2n}} \leq \sigma^2 \leq \frac{2n\hat{\sigma}^2}{\chi^2_{1-\alpha/2,2n}}) &= 1-\alpha \\
P(\sqrt{\frac{2n\hat{\sigma}^2}{\chi^2_{\alpha/2,2n}}} \leq \sigma \leq \sqrt{\frac{2n\hat{\sigma}^2}{\chi^2_{1-\alpha/2,2n}}}) &= 1-\alpha
\end{aligned}$$

With $n = 50$, $\hat{\sigma} = 12.7$ and $\alpha = 0.10$ which gives $\chi^2_{1-\alpha/2,2n} = \chi^2_{0.95,100} = 77.929$ and $\chi^2_{\alpha/2,2n} = \chi^2_{0.05,16} = 124.342$ we get: $\left[\sqrt{\frac{2\cdot 50\cdot 12.7^2}{124.342}},\ \sqrt{\frac{2\cdot 50\cdot 12.7^2}{77.929}}\right] = \underline{\underline{[11.4,\ \ 14.4]}}$

Since we have a large n the two intervals are almost equal. As the intervals contain 14 we can not from this claim that Hanna is better.