University of Stavanger

EXAM IN: STA500 INTRODUCTION TO PROBABILITY AND STATISTICS 2

DURATION: 4 HOURS DATE: FEBRUARY 9, 2015

PERMITTED AIDS: Approved simple calculator (HP30S, Casio FX82, TI-30, Citizen SR-270X, Texas BA II Plus or HP17bII+).
One yellow A4 size sheet with your own handwritten notes.

THE EXAM CONSISTS OF 2 PROBLEMS ON 4 PAGES, 9 PAGES OF ENCLOSURES.

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Problem 1:

A factory is producing cable, and now and then there is a failure on the produced cable. Let X denote the distance (in kilometers) along the cable between two consecutive failures. We shall assume that the failures occur independently of each others, that is, the lengths of consecutive distances between failures, $X_1, X_2, X_3 \ldots$, are independent random variables.

By experience the distribution of the distance between consecutive failures is well described by a distribution with cumulative distribution function

$$F(x) = 1 - e^{-x^3/\theta} \quad \text{for } x \geq 0 \text{ and } \theta > 0.$$

Assume first that $\theta = 500$.

a) Show that $P(X > 10) = 0.135$.
Calculate $P(5 \leq X \leq 10)$.
If we have observed that it is 5 kilometers since the previous failure, what is the probability that there will be at least 5 new kilometers to the next failure?

A customer wants to buy 10 kilometers of continuous cable with no failures. Let Z be the number of distances between consecutive failures (i.e. the number of X_is) which needs to be examined until the first time one finds a distance which is at least 10 kilometers.

b) Which distribution does Z follow? Explain.
Calculate $P(Z > 3)$ and $\text{E}(Z)$.

The factory is not satisfied with the quality of their product, and they make some changes in the production process. We assume that the distribution of the distances between failures after the changes still is of the same type as specified in the beginning of the problem, but now with a new unknown value of θ.

c) Show that the maximum likelihood estimator for θ, based on n independent measurements $X_1, X_2, \ldots, X_n$, is:

$$\hat{\theta} = \frac{\sum_{i=1}^{n} X_i^3}{n}$$

d) Explain why $Y = 2X^3/\theta$ has a χ^2_2-distribution. (Hint: Which type of distribution does X have?)
Use this to explain that $\frac{2n}{\theta}\hat{\theta}$ has a χ^2_{2n}-distribution.
Examine whether $\hat{\theta}$ is unbiased.

The factory has measured 8 consecutive distances between failures after the changes in the production process. The measured data are given below:

15.1	17.4	14.0	16.6	15.5	13.3	12.1	11.9

e) Calculate the estimate of θ from the data above.
Use the information from d) as starting point and derive an exact 95% confidence interval for θ.
Also find the 95% Wald confidence interval for θ.
Compare the two intervals and comment.

Problem 2:

The arrival of customers to a customer service counter at a large sporting goods store is described by a Poisson process with rate λ per minute. The time it takes to serve each customer is exponentially distributed with rate $\gamma = 0.25$ per minute. There are c staff serving customers at the service counter. The service times are independent of each other and independent of the arrival process.

a) Consider a situation with $\lambda = 0.4$.
What is the probability that exactly 20 customers arrive during one hour?
What is the probability that less than 20 customers arrive during one hour?
What is the expected time from we start observing until 20 customers have arrived?

b) How many staff, c, need to serve customers for the queue to be stable? Find an expression for c in terms of λ. What is the smallest number of staff required for a stable queue when $\lambda = 0.4$?
When you arrive at the service counter you are customer number 3 in the waiting line and there are $c = 3$ staff serving customers (i.e. 3 customers are being served and 2 are waiting in line before you).
What is the expected time until you start being served?
What is the probability that you have to wait more than 5 minutes until you start being served?

Let $X(t)$ be the total number of customers in the system (being served or waiting in queue) at time t. We consider a situation with $c = 2$ staff serving customers and $\lambda = 0.4$.

c) Explain briefly what type of process $\{X(t) : t \geq 0\}$ is.
Draw the transition graph for the process and write the transition rates on the plot. Include at least the first five states (the states 0 to 4) in your plot.
Set up the steady state equations for the first four states (the states 0 to 3).
(You do not need to solve the equations.)

In reality not all customers that arrive at the service counter for help choose to stay in the queue. Assume that the probability that a customer chooses to join the queue when there are k customers in the waiting line is $1 - k/5$ for $k = 0, 1, 2, \ldots, 4$ and 0 when there are 5 customers in the waiting line.

Furthermore customers may also leave the queue. Assume that each customer waiting in the queue leave the queue with constant rate $\alpha = 0.1$ (independent of the service times and the arrival process).

Let as before $X(t)$ be the total number of customers in the system (being served or waiting in queue), $c = 2$ and $\lambda = 0.4$. It can be shown that with the given specifications the steady state probabilities for the process are: $\pi_0 = 0.190$, $\pi_1 = 0.304$, $\pi_2 = 0.243$, $\pi_3 = 0.162$, $\pi_4 = 0.074$, $\pi_5 = 0.022$, $\pi_6 = 0.004$ and $\pi_7 = 0.0003$.

d) Draw the transition graph for the process and write the transition rates on the plot.
Calculate the expected number of customers waiting in queue.
If we also had calculated the expected number of customers waiting in queue in point c), would that number have been higher or lower than the expected number we got now? Explain briefly without doing any calculations.