Solution exam February 9, 2015

Problem 1:

a)

$$\begin{aligned} P(X > 10) &= 1 - P(X \le 10) = 1 - (1 - e^{-10^3/500}) = \underline{\underline{0.135}} \\ P(5 \le X \le 10) &= F_X(10) - F_X(5) = 1 - e^{-10^3/500} - (1 - e^{-5^3/500}) = \underline{\underline{0.643}} \\ P(X > 10 | X > 5) &= \frac{P((X > 10) \cap (X > 5))}{P(X > 5)} = \frac{P(X > 10)}{P(X > 5)} = \frac{e^{-10^3/500}}{e^{-5^3/500}} = \underline{\underline{0.174}} \end{aligned}$$

b) The situation is characterised by:

- We check "success" or not "success" in each trial - whether the distance between two failures is a least 10 km or not.
- The probability of "success" is the same in all trials - $p = 0.135$ that the distance is at least 10 km.
- Independent trials - independent lengths of distances between failures.
- The trials continue until the first "successes" - measure distances between failures until first time the distance is at least 10 km.

Z="the number of distances between consecutive failures which needs to be examined until the first time one finds a distance which is at least 10 kilometres" is then having a geometric distribution with $p = 0.135$ (a special case of the negative distribution where we look at the number of trials until the first success).

$$\begin{aligned} P(Z > 3) &= 1 - P(Z \le 3) = 1 - (P(Z = 1) + P(Z = 2) + P(Z = 3)) \\ &= 1 - 0.135 - 0.135 \cdot (1 - 0.135) - 0.135 \cdot (1 - 0.135)^2 = \underline{\underline{0.647}} \\ \mathrm{E}(Y) &= \frac{1}{p} = \frac{1}{0.135} = \underline{\underline{7.4}} \end{aligned}$$

c)

$$f(x) = \frac{dF(x)}{dx} = \frac{d}{dx}[1 - e^{-x^3/\theta}] = \frac{3x^2}{\theta} e^{-x^3/\theta}$$

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} f(x_i; \theta) = \prod_{i=1}^{n} \frac{3x_i^2}{\theta} e^{-x_i^3/\theta} = \frac{3^n \prod_{i=1}^n x_i^2}{\theta^n} e^{-\sum_{i=1}^n x_i^3/\theta} \\ l(\theta) &= \ln L(\theta) = \ln(3^n) + \ln(\prod_{i=1}^{n} x_i^2) - \ln(\theta^n) + \ln(e^{-\sum_{i=1}^n x_i^3/\theta}) \\ &= n\ln(3) + \ln(\prod_{i=1}^{n} x_i^2) - n\ln(\theta) - \sum_{i=1}^{n} x_i^3/\theta \\ \frac{\partial l(\theta)}{\partial \theta} &= -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} x_i^3 = 0 \quad \Rightarrow \quad n\theta = \sum_{i=1}^{n} x_i^3 \end{aligned}$$

I.e. the MLE becomes: $\underline{\underline{\hat{\theta} = \frac{1}{n} \sum_{i=1}^n X_i^3}}$.

d) If we first look at X and compare the pdf of X calculated in c) with the pdf of the Weibull distribution $f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}$ we see that X has a Weibull distribution with parameters $\alpha = 1/\theta$ and $\beta = 3$. We can then use the transformation results which says that if X has a Weibull distribution with parameters α and β, then $Y = 2\alpha X^\beta$ has a χ^2_2-distribution. Here that implies that $Y = 2X^3/\theta$ has a χ^2_2-distribution.

$$\frac{2n\hat{\theta}}{\theta} = \frac{2}{\theta} \sum_{i=1}^{n} X_i^3 = \sum_{i=1}^{n} \frac{2X_i^3}{\theta} = \sum_{i=1}^{n} Y_i$$

Since we have that a sum of independent χ^2-distributed variables is χ^2-distributed with parameter ("degrees of freedom") equal to the sum of the parameters of each variable (collection of formulas), we have here (since all the Y_i are independent and χ^2-distributed) that $\frac{2n\hat{\theta}}{\theta} = \sum_{i=1}^n Y_i$ is $\underline{\underline{\chi^2_{2n}\text{-distributed}}}$.

Recall that if a variable, V, is χ^2_ν-distributed, then $\mathrm{E}(V) = \nu$. We can use this to calculate $\mathrm{E}(\hat{\theta})$:

$$\mathrm{E}(\hat{\theta}) = \frac{\theta}{2n} \mathrm{E}(\frac{2n}{\theta}\hat{\theta}) = \frac{\theta}{2n} 2n = \theta \quad \text{i.e. the estimator is } \underline{\underline{\text{unbiased}}}!$$

e) $\hat{\theta} = \frac{\sum_{i=1}^8 x_i^3}{8} = \frac{25562.5}{8} = \underline{\underline{3195}}$. Confidence interval:

$$\begin{aligned} P(\chi^2_{1-\alpha/2,2n} \le \frac{2n\hat{\theta}}{\theta} \le \chi^2_{\alpha/2,2n}) &= 1 - \alpha \\ P(\frac{\chi^2_{1-\alpha/2,2n}}{2n\hat{\theta}} \le \frac{1}{\theta} \le \frac{\chi^2_{\alpha/2,2n}}{2n\hat{\theta}}) &= 1 - \alpha \\ P(\frac{2n\hat{\theta}}{\chi^2_{1-\alpha/2,2n}} \ge \theta \ge \frac{2n\hat{\theta}}{\chi^2_{\alpha/2,2n}}) &= 1 - \alpha \\ P(\frac{2n\hat{\theta}}{\chi^2_{\alpha/2,2n}} \le \theta \le \frac{2n\hat{\theta}}{\chi^2_{1-\alpha/2,2n}}) &= 1 - \alpha \end{aligned}$$

I.e. a $(1-\alpha)100\%$ confidence interval for θ is given by:

$$\underline{\underline{\left[\frac{2n\hat{\theta}}{\chi^2_{\alpha/2,2n}},\ \frac{2n\hat{\theta}}{\chi^2_{1-\alpha/2,2n}}\right]}}$$

With $n = 8$, $\hat{\theta} = 3195$ and $\alpha = 0.05$ which gives $\chi^2_{1-\alpha/2,2n} = \chi^2_{0.975,16} = 6.908$ and $\chi^2_{\alpha/2,2n} = \chi^2_{0.025,16} = 28.845$ we get:

$$\left[\frac{2\cdot 8\cdot 3195}{28.845},\ \frac{2\cdot 8\cdot 3195}{6.908}\right] = \underline{\underline{[1772,\ \ 7400]}}$$

We start by finding the second derivative of the log-likelihood at $\hat{\theta}$:

$$\begin{aligned}
\frac{\partial l(\theta)}{\partial\theta} &= -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i^3 \\
J(\theta) &= \frac{\partial^2 l(\theta)}{\partial\theta^2} = \frac{n}{\theta^2} - 2\frac{\sum_{i=1}^{n} x_i^3}{\theta^3} \\
J(\hat{\theta}) &= \frac{n}{\hat{\theta}^2} - 2n\frac{\sum_{i=1}^{n} x_i^3}{n}\frac{1}{\hat{\theta}^3} = \frac{n}{\hat{\theta}^2} - 2n\frac{1}{\hat{\theta}^2} = -\frac{n}{\hat{\theta}^2}
\end{aligned}$$

The Wald confidence interval then becomes:

$$\left[\hat{\theta} - z_{\alpha/2}\sqrt{-1/J(\hat{\theta})}, \hat{\theta} + z_{\alpha/2}\sqrt{-1/J(\hat{\theta})}\right] = \left[\hat{\theta} - z_{\alpha/2}\sqrt{\hat{\theta}^2/n}, \hat{\theta} + z_{\alpha/2}\sqrt{\hat{\theta}^2/n}\right]$$

and inserting $z_{0.025} = 1.96$, $n = 8$ and $\hat{\theta} = 3195$ we get:

$$[3195 - 1.96\sqrt{3195^2/8},\ \ 3195 + 1.96\sqrt{3195^2/8}] = \underline{\underline{[981,\ \ 5409]}}$$

Comparing this with the exact interval we see that the Wald interval differs quite a bit. The Wald interval is only an approximate interval, where the approximation is better the more observations we have. With only 8 observations the approximation does not seem to be very good.

Problem 2:

a) The number of customers during $t = 60$ minutes, Y, is Poisson-distributed with expectation $\lambda t = 0.4 \cdot 60 = 24$.

$$\begin{aligned}
P(Y = 20) &= \frac{24^{20}}{20!}e^{-24} = \underline{\underline{0.062}} \\
P(Y < 20) &= P(Y \le 19) \approx P(Z \le \frac{19 + 0.5 - \mathrm{E}(X)}{\sqrt{\mathrm{Var}(X)}}) \\
&= P(Z \le \frac{19 + 0.5 - 24}{\sqrt{24}}) = P(Z \le -0.92) = \underline{\underline{0.18}}
\end{aligned}$$

The distribution of the time until event number 20 in a Poisson process with $\lambda = 0.4$ has a gamma distribution with parameters $\alpha = 20$ and $\beta = 1/0.4 = 2.5$. The expected time is then $\alpha\beta = 20 \cdot 2.5 = \underline{\underline{50}}$.

b) The queue is stable as long as the rate of arrivals to the queue is smaller than the maximum rate of departures from the queue. The arrival rate is λ and with c staff the maximum departure rate is $c\cdot\gamma$. I.e. the queue is stable when $\lambda < c\cdot\gamma$, implying that we need to have $c > \lambda/\gamma = \lambda/0.25 = \underline{\underline{4\lambda}}$ staff. With $\lambda = 0.4$ we need $c > 4 \cdot 0.4 = 1.6$, i.e. at least $\underline{\underline{2}}$ staff.

Let $W_3 = Z_1 + Z_2 + Z_3$ denote the waiting time until you start being served. Here Z_i denotes the times between each time the queue moves (i.e. a customer is finished), and the queue needs to move three times before you start being served. Further due to the memoryless property of the exponential distribution $Z_i = \min\{V_1, V_2, V_3\}$ where V_1, V_2, V_3 have the same distribution as the service time, i.e. they are iid exponential with $\mathrm{E}(V_j) = 1/0.25 = 4$ and Z_i is thus exponential with $\mathrm{E}(Z_i) = \mathrm{E}(V_j)/3 = 4/3$. Then:

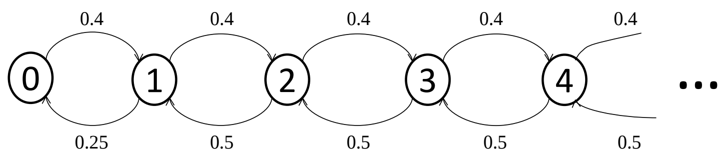
$$\mathrm{E}(W_3) = \mathrm{E}(Z_1) + \mathrm{E}(Z_2) + \mathrm{E}(Z_3) = 3 \cdot 4/3 = \underline{\underline{4\ \text{minutes}}}$$

If we look at when customers are finished, we have explained above that the times between when customers are finished is exponentially distributed with expectation 4/3. This means that the process where we record when customers are finished is a Poisson process with rate $\lambda = 1/(4/3) = 0.75$. If we let Y be the number of customers who finish during 5 minutes we have that this number is Poisson distributed with parameter $\lambda t = 0.75 \cdot 5 = 3.75$. If it takes more than 5 minutes before you (customer number 3) start being served this means that less than 3 customers are finished during the 5 minutes. I.e.:

$$\begin{aligned}
P(W_3 > 5) &= P(Y < 3) = P(Y = 0) + P(Y = 1) + P(Y = 2) \\
&= \frac{3.75^0}{0!}e^{-3.75} + \frac{3.75^1}{1!}e^{-3.75} + \frac{3.75^2}{2!}e^{-3.75} = \underline{\underline{0.277}}
\end{aligned}$$

c) The process is a continuous time Markov chain of the type called <u>birth and death process.</u> This is because all events happens in continuous time, with independent and exponentially distributed times until next arrival (since arrivals follow an Poisson process) and next departure (since service times are exponential). The memoryless property of the exponential distribution implies the Markov property, and with continuous time two or more events will not happen at the same time - i.e. the process wil only move up or down in steps of one.

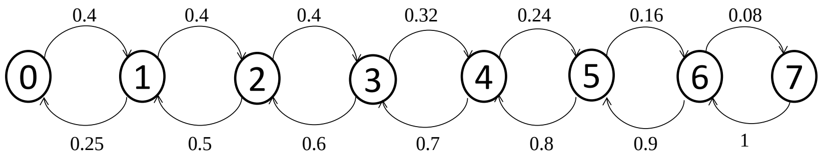
The transition graph is displayed in the figure below. Balancing the rate out and the rate in in



each state give the following steady state equations for the first four states:

$$\begin{aligned}
0:&\quad \pi_1 \cdot 0.25 = \pi_0 \cdot 0.4 \\
1:&\quad \pi_0 \cdot 0.4 + \pi_2 \cdot 0.5 = \pi_1 \cdot 0.25 + \pi_1 \cdot 0.4 = \pi_1 \cdot 0.65 \\
2:&\quad \pi_1 \cdot 0.4 + \pi_3 \cdot 0.5 = \pi_2 \cdot 0.5 + \pi_2 \cdot 0.4 = \pi_2 \cdot 0.9 \\
3:&\quad \pi_2 \cdot 0.4 + \pi_4 \cdot 0.5 = \pi_3 \cdot 0.5 + \pi_3 \cdot 0.4 = \pi_3 \cdot 0.9
\end{aligned}$$

d) The transition graph is displayed in the figure below. Notice that when there are k customers



waiting in line the birth rate will be $(1-k/5)\lambda$ (since only a portion $(1-k/5)$ of the customers join the queue), implying e.g. that the rate in state 4 (with 2 waiting in line) is $(1-2/5)\cdot 0.4 = 0.24$ etc. The death rates when there are k customers waiting in line is $0.5 + k \cdot 0.1$ because all k waiting customers may leave the queue with rate 0.1.

The expected number of customers *waiting in the queue* (notice that there are no customers waiting in queue in state 0, 1 and 2) is:

$$0\cdot(\pi_0+\pi_1+\pi_2)+1\cdot\pi_3+2\cdot\pi_4+3\cdot\pi_5+4\cdot\pi_6+5\cdot\pi_7 = 0.162+2\cdot 0.074+3\cdot 0.022+4\cdot 0.004+5\cdot 0.0003 = \underline{\underline{0.39}}$$

In point c) the birth rates are higher (all arriving customers join the queue) and the death rates are lower (customers do not leave the queue without being served). I.e. the process will be in the higher states more of the time, and there is also no upper limit on the queue, thus the expected number of customers waiting in queue will be higher in point c).