University of Stavanger

EXAM IN: STA500 INTRODUCTION TO PROBABILITY AND STATISTICS 2

DURATION: 4 HOURS DATE: Feburary 15, 2017

PERMITTED AIDS: Approved simple calculator (HP30S, Casio FX82, TI-30, Citizen SR-270X, Texas BA II Plus or HP17bII+).

One yellow A4 size sheet with handwritten notes is allowed.

Both sides of the sheet can be used.

THE EXAM CONSISTS OF 7 PROBLEMS ON 3 PAGES, 9 PAGES OF ENCLOSURES.

COURSE RESPONSIBLE: Jörn Schulz PHONE:

Problem 1: Let $X_1, \dots, X_n$ be independent and identically distributed random variables with density

$$f(x) = \begin{cases} \frac{4}{\theta}x^3 \exp(-\frac{1}{\theta}x^4), & x > 0 \\ 0, & x \leq 0 \end{cases} \quad (\theta > 0).$$

a) Show that the MLE for θ becomes $\hat{\theta} = \frac{1}{n}\sum_{i=1}^n x_j^4$. Show that $\hat{\theta}$ is a maximum.

b) Find the cumulative distribution function for X_i.
Moreover, show that $X_i^4 \sim Exp(\theta)$.
Is $\hat{\theta}$ an unbiased estimator of θ?

Problem 2: Suppose we observe birds of prey. We assume that the waiting time (in hours) until we observe the next bird is independent exponentially distributed with unknown mean parameter $\beta > 0$. We have observed a sample of size $n = 10$, with waiting times

2.53, ,1.73, 3.99, 0.09, 0.17, 0.95, 0.94, 0.44, 8.18, 0.09.

The ML estimator for β is given by $\hat{\beta} = \frac{1}{n}\sum_{i=1}^n x_i$ and $J(\hat{\beta}) = -\frac{n}{\hat{\beta}^2}$ (don't show that).

a) Derive an exact 95% confidence interval for β.
Construct a 95%-Wald-confidence interval for β.
Suppose we observe another sample of size $n = 20$ with exactly the same value for $\hat{\beta}$ as before, i.e., $\hat{\beta}$ is the same for both sample sizes. What is the 95%-Wald-confidence interval in this case?
Compare the three intervals and comment!

Problem 3: Assume that you want to investigate the proportion θ of defective items manufactured at a production line. Your colleague takes a random sample of 30 independent items. Three were defective in the sample. Let X be the number of defective items.

a) What is the distribution of X? Explain your answer!
Assume a Beta distribution for the prior with known parameters α and β, i.e., $\theta \sim Beta(\alpha, \beta)$ with prior-density $p(\theta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$. Compute the posterior of θ. What kind of distribution has the posterior?
What is the Bayes estimator for θ?

Suppose your colleague now tells you that he did not decide on the sample size before the sampling was performed. His sampling plan was to keep on sampling items until he had found three defective ones. It just happen that the $30'th$ item was the third one to be defective.

b) Redo the posterior calculation, this time under the new sampling scheme.
Discuss the result in comparison with a).

Problem 4: Consider the Markov chain model $\{X_n,\ n = 0, 1, \ldots\}$ with state space $\mathcal{S} = \{0, 1, 2\}$ and transition probability matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha & 0 \\ \frac{\alpha}{2} & 1-\alpha & \frac{\alpha}{2} \\ 0 & \alpha & 1-\alpha \end{pmatrix}, \ 0 < \alpha \leq 1.$$

a) What is $P(X_4 = 1 | X_3 = 1, X_1 = 0)$
For which $\alpha \in (0, 1]$ is X_n a-periodic? Explain your answer!
For which $\alpha \in (0, 1]$ is X_n periodic and what is the period? Explain your answer!

b) When do we have steady state probabilities? Why? Calculate the steady state probabilities!
How do we interpret the matrix P^2?
Given $\alpha = 1$, calculate p^2_{11} and $P(X_6 = 0, X_4 = 0 | X_3 = 1, X_1 = 2)$.

Problem 5: Let $X_1, \ldots, X_n, n \in \mathbb{N}$, independent and identically distributed random variables with $E(X_i) = \mu \in \mathbb{R}$ and $Var(X_i) = \sigma^2 > 0$.

a) Show that

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \text{and} \qquad \tilde{\mu} = \sum_{i=1}^{n} w_i X_i,\ w_i = \frac{2i}{n(n+1)}$$

are unbiased estimators for μ. Which one is the better estimator? Proof your answer.

Problem 6: An auto manufacturer has certain requirements for the automobiles that it manufactures. The gasoline consumption is normally distributed with mean $\mu = 5$ and standard deviation $\sigma = \frac{4}{5}$ in l/100km (liters per 100km) as the cars come off the assembly line. The manufacturer then tests the gasoline consumption and re-manufactures any unit that tests above 6 l/100km.

a) What fraction of the manufacturer's automobiles are likely to be re-manufactured?
If the company improves quality control, they can reduce the value of σ. What value of σ will ensure that not more than 1 percent of the automobiles has to be re-manufactured?

Problem 7: A small petrol filling station has two pumps that can serve two cars at the same time independent of each other. In addition, there is space for three further waiting spaces. Vehicles have to pass by the petrol station if all waiting spaces are occupied. Vehicles arrive at random (a Poisson Process) with rate $\lambda > 0$, and the service times (in minutes) are exponential distributed with rate $\gamma > 0$.

a) If the number of vehicles at the station is used to represent the state of the system, write down a state transition graph for the system including the rates.
Show that in steady state, the probability that there are k vehicles at the station is given by

$$\pi_0 = \frac{1-\rho}{1+\rho-2\rho^6} \quad \text{and} \quad \pi_k = 2\rho^k \pi_0,\ k = 1, 2, 3, 4, 5$$

where $\rho = \frac{\lambda}{2\gamma}$.
Given an arrival rate of $\lambda = 0.4$ and an expected service time of 4 minutes, find the mean number of vehicles at the station.

Solutions

1.a
The likelihood is given by

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} f(\theta; x_i) = \prod_{i=1}^{n} \frac{4}{\theta} x_i^3 \exp(-\frac{1}{\theta} x_i^4) \\ &= \frac{4^n}{\theta^n} \prod_{i=1}^{n} x_i^3 \exp(-\frac{1}{\theta} x_i^4) \\ &= \frac{4^n}{\theta^n} \exp\left(-\frac{1}{\theta} \sum_{i=1}^{n} x_i^4\right) \prod_{i=1}^{n} x_i^3 \end{aligned}$$

and the log-likelihood by

$$\begin{aligned} l(\theta) &= \ln L(\theta) = n \ln\left(\frac{4}{\theta}\right) - \frac{1}{\theta} \sum_{i=1}^{n} x_i^4 + \sum_{i=1}^{n} \ln(x_i^3) \\ &= n \ln(4) - n \ln(\theta) - \frac{1}{\theta} \sum_{i=1}^{n} x_i^4 + 3 \sum_{i=1}^{n} \ln(x_i). \end{aligned}$$

This leads to

$$\begin{aligned} \frac{d}{d\theta} l(\theta) &= -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} x_i^4 = -\frac{n}{\theta^2}\left(\theta + \frac{1}{n} \sum_{i=1}^{n} x_i^4\right) = 0 \\ &\Leftrightarrow \hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} x_i^4. \end{aligned}$$

$\hat{\theta}$ is a maximum because

$$J(\theta) = \frac{d^2}{d\theta^2} l(\lambda) = \frac{n}{\theta^2} - \frac{2}{\theta^3} \sum_{i=1}^{n} x_i^4 = \frac{n}{\theta^2} - \frac{2}{\theta^3} n\hat{\theta}$$

and

$$J(\hat{\theta}) = \frac{n}{\hat{\theta}^2} - \frac{2}{\hat{\theta}^3} n\hat{\theta} = \frac{1}{\hat{\theta}^2}\left(n - \frac{2}{\hat{\theta}} n\hat{\theta}\right) = -\frac{n}{\hat{\theta}^2} < 0.$$

Thus, $\hat{\theta}$ is a maximizer and therewith $\hat{\theta}$ is a MLE.

1.b
For $z > 0$ we have

$$\begin{aligned} F_X(z) &= P(X \leq z) = \int_0^z \frac{4}{\theta} x^3 \exp\left(-\frac{1}{\theta} x^4\right) dx = \left[-\exp\left(-\frac{1}{\theta} x^4\right)\right]_0^z \\ &= 1 - \exp\left(-\frac{1}{\theta} z^4\right) \end{aligned}$$

and therefore

$$F_{X^4}(z) = P(X^4 \le z) = P(X \le z^{\frac{1}{4}}) = 1 - \exp\left(-\frac{1}{\theta} z\right).$$

This is the cumulative distribution function of the Exponential distribution with parameter $\beta = \theta$, i.e., $X^4 \sim Exp(\theta)$. The estimator $\hat{\theta}$ is unbiased because of

$$E(\hat{\theta}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i^4\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i^4) = E(X_i^4) = \theta.$$

2.a

The density of the exponential function parameterized by β is $f(x) = \frac{1}{\beta}\exp(-\frac{1}{\beta}x),\ x \ge 0$. The ML estimator of β is given by $\hat{\beta} = \bar{x} = 1.911$ given the 10 observations.

The exact CI can be calculated from the transformation formulas. We have for the exponential distribution that $2X_i/\beta$ has a χ_2^2-distribution if X_i is exponential distributed with parameter β. Furthermore, we have

$$\sum \frac{2X_i}{\beta} = \frac{2n}{\beta}\frac{1}{n}\sum X_i = \frac{2n}{\beta}\hat{\beta} \sim \chi^2_{2n}.$$

Therefore, the CI is given by

$$\begin{aligned}
P(\chi^2_{1-\alpha/2,2n} \le \frac{2n}{\beta}\hat{\beta} \le \chi^2_{\alpha/2,2n}) &= 1-\alpha \\
\Leftrightarrow P\left(\frac{\chi^2_{1-\alpha/2,2n}}{2n\hat{\beta}} \le \frac{1}{\beta} \le \frac{\chi^2_{\alpha/2,2n}}{2n\hat{\beta}}\right) &= 1-\alpha \\
\Leftrightarrow \left(\frac{2n\hat{\beta}}{\chi^2_{\alpha/2,2n}} \le \beta \le \frac{2n\hat{\beta}}{\chi^2_{1-\alpha/2,2n}}\right) &= 1-\alpha.
\end{aligned}$$

Given $n = 10$, $\hat{\beta} = \bar{x} = 1.911$, $\alpha = 0.05$, we have the exact CI

$$\left[\frac{2\cdot 10\cdot 1.911}{34.170}, \frac{2\cdot 10\cdot 1.911}{9.591}\right] = [1.118, 3.985].$$

We know that $J(\hat{\theta}) = -\frac{n}{\hat{\beta}^2}$ which leads to the Wald interval

$$\left(\hat{\beta} \pm z_{\alpha/2}\sqrt{-\frac{1}{J(\hat{\beta})}}\right) = (1.911 \pm 1.96\cdot 0.604) = (0.727, 3.094).$$

In case of $n = 20$ and identical $\bar{x}$, the Wald-interval is

$$(1.911 \pm 1.96\cdot 0.427) = (1.074, 2.748).$$

We observe that the exact CI is larger than the Wald-CI for $n = 10$. This might be surprising but can be explained by the fact that we were lucky with our sample which led to a relative small estimated variance. However, we see that we underestimate the the exact CI by the Wald-CI. Thus, we conclude that we need a larger sample size. Moreover, as expected, we see that the 2nd Wald-CI for $n = 20$ is smaller than for $n = 10$.

3.a
All 30 trials are independent. We have either no success or success (defective or non-defective), the probability of a defective item is the same in each trial, and we have a specified number of trials, namely $n = 30$. Thus, the data is binomial distributed and the likelihood is

$$L(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x} \propto \theta^x(1-\theta)^{n-x}$$

and the prior

$$p(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}.$$

The posterior density is therewith

$$p(\theta|x) \propto \theta^x(1-\theta)^{n-x}\theta^{\alpha-1}(1-\theta)^{\beta-1} = \theta^{(x+\alpha)-1}(1-\theta)^{(n-x+\beta)-1},$$

which we recognize as being proportional to Beta density, i.e., the posterior distribution is $Beta(x+\alpha, n-x+\beta) = Beta(3+\alpha, 27+\beta)$.
The Bayes estimator is $\hat{\theta}_{Bayes} = \frac{3+\alpha}{30+\alpha+\beta}$.

3.b
Now, we have repeated trials until $k = 3$ defectives with the $30'th$ item was the third one to be defective. Therefore, the likelihood function is now given by the negative binomial distribution, i.e.,

$$L(x|\theta) = \binom{x-1}{k-1}\theta^k(1-\theta)^{x-k} \propto \theta^k(1-\theta)^{x-k}$$

with $k = 3$ and $x = 30$.

Assume that you want to investigate the proportion θ of defective items manufactured at a production line. Your colleague takes a random sample of 30 items. Three were defective in the sample. Let X be the number of defective items. The posterior density is therewith

$$p(\theta|x) \propto \theta^k(1-\theta)^{x-k}\theta^{\alpha-1}(1-\theta)^{\beta-1} = \theta^{(k+\alpha)-1}(1-\theta)^{(x-k+\beta)-1},$$

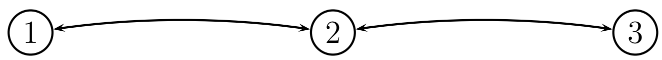
i.e., we have again $Beta(3+\alpha, 27+\beta)$ distribution as a posterior distribution. In Bayesian analysis we are looking at the posterior, where the data is fixed. Therefore, we get the same posterior as in a) even if a different variable is stochastic in the likelihood. It is only the shape of the likelihood that matter for inference.

4.a
$P(X_4 = 1|X_3 = 1, X_1 = 0) = P(X_4 = 1|X_3 = 1) = p_{11} = 1 - \alpha.$

It is given that $0 < \alpha \leq 1$. For $\alpha = 1$, the transition probability matrix becomes

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

and the transition graph

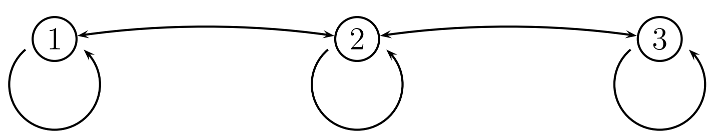


and therefore the Markov chain X_n is periodic with period 2. For example, if the Markov chain is in state 1, than we are in two steps in state 1 again.

In case of $\alpha \in (0,1)$, i.e. $0 < \alpha < 1$, we have that $(1-\alpha) > 0$, $\alpha > 0$ and $\frac{\alpha}{2} > 0$. Thus, except $p_{02} = 0$ and $p_{20} = 0$ all elements of P have a positive probability,

$$P = \begin{pmatrix} 1-\alpha & \alpha & 0 \\ \frac{\alpha}{2} & 1-\alpha & \frac{\alpha}{2} \\ 0 & \alpha & 1-\alpha \end{pmatrix}.$$

Thus, the transition matrix can be written by



Therewith, the Markov chain X_n is aperiodic for all $\alpha \in (0,1)$.

4.b
In case of $0 < \alpha < 1$, we have steady state probabilities because the Markov chain is irreducible, aperiodic and has a finite state space. The steady probabilities for

$0 < \alpha < 1$ are

$$\begin{aligned}
& \Pi = P^T \Pi \\
\Leftrightarrow \quad & \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 1-\alpha & \frac{1}{2}\alpha & 0 \\ \alpha & 1-\alpha & \alpha \\ 0 & \frac{1}{2}\alpha & 1-\alpha \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} \\
\Leftrightarrow \quad & \pi_0 = (1-\alpha)\pi_0 + \frac{1}{2}\alpha\pi_1 \\
& \pi_2 = \frac{1}{2}\alpha\pi_1 + (1-\alpha)\pi_2 \\
& \text{plus the condition } \pi_1 = 1 - \pi_0 - \pi_2 \\
\Leftrightarrow \quad & \pi_0 = \pi_0 - (1-\alpha)\pi_0 = \alpha\pi_0 = \frac{1}{2}\alpha\pi_1 \Leftrightarrow \pi_0 = \frac{1}{2}\pi_1 \\
& \pi_2 = \pi_2 - (1-\alpha)\pi_2 = \alpha\pi_2 = \frac{1}{2}\alpha\pi_1 \Leftrightarrow \pi_2 = \frac{1}{2}\pi_1 \\
& \pi_1 = 1 - \frac{1}{2}\pi_1 - \frac{1}{2}\pi_1 \Leftrightarrow 2\pi_1 = 1 \Leftrightarrow \pi_1 = \frac{1}{2}.
\end{aligned}$$

Therefore, the steady state probabilities are

$$\begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.5 \\ 0.25 \end{pmatrix}.$$

The matrix P^2 contains all transition probabilities of the homogeneous Markov chain $\{X_n, \ n = 0, 1, \dots\}$ for two steps transitions. More precise, each entry $p_{ij}^2 = P(X_{n+2} = j \mid X_n = i)$ define the conditional probability that the Markov chain will be in state j at time $n+2$ if the chain is in state i at time n. Given $\alpha = 1$, the matrix P^2 is

$$P^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Therefore, $p_{11}^2 = 1$ and

$$\begin{aligned}
P(X_6 = 0, X_4 = 0 | X_3 = 1, X_1 = 2) &= P(X_6 = 0, X_4 = 0 | X_3 = 1) \\
&= P(X_6 = 0 | X_4 = 0) P(X_4 = 0 | X_3 = 1) \\
&= p_{00}^2 p_{10} = 0.5 \cdot 0.5 = 0.25.
\end{aligned}$$

5.a

We have

$$E(\hat{\mu}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n} n\mu = \mu$$

and

$$E(\tilde{\mu}) = E\left(\sum_{i=1}^{n} \frac{2i}{n(n+1)} X_i\right) = \frac{2}{n(n+1)} \sum_{i=1}^{n} iE(X_i) = \frac{2}{n(n+1)} \mu \sum_{i=1}^{n} i = \mu$$

with $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ (see collection of formulas), i.e., $\hat{\mu}$ and $\tilde{\mu}$ are unbiased. In order to find the best estimator, we calculate the variance by

$$Var(\hat{\mu}) = Var\left(\frac{1}{n}\sum_{i=1}^n X_i\right) == \frac{1}{n^2}\sum_{i=1}^n Var(X_i) = \frac{1}{n^2}n\sigma^2 = \frac{1}{n}\sigma^2$$

and

$$\begin{aligned} Var(\tilde{\mu}) &= Var\left(\sum_{i=1}^n \frac{2i}{n(n+1)}X_i\right) = \frac{4}{n^2(n+1)^2}Var\left(\sum_{i=1}^n iX_i\right) \\ &= \frac{4}{n^2(n+1)^2}\sum_{i=1}^n i^2 Var(X_i) = \frac{4}{n^2(n+1)^2}\sigma^2\sum_{i=1}^n i^2 \\ &= \frac{4}{n^2(n+1)^2}\frac{n(n+1)(2n+1)}{6}\sigma^2 = \frac{2}{3}\frac{2n+1}{n(n+1)}\sigma^2. \end{aligned}$$

Now, we have

$$\frac{2}{3}\frac{2n+1}{n(n+1)} = \frac{1}{n}\left(\frac{2}{3}\frac{2n+1}{n+1}\right) \geq \frac{1}{n}, \forall n \geq 1.$$

Therefore, $Var(\hat{\mu}) \leq Var(\tilde{\mu}), \forall \mu \in \mathbb{R}$, i.e., $\hat{\mu}$ is a better estimator than $\tilde{\mu}$.

6.a

We have $X \sim N(5, 0.8^2)$. Therefore, calculations are as follows:

$$\begin{aligned} P(X > 6) &= 1 - P(X \leq 6) = 1 - P\left(\frac{X-5}{0.8} \leq \frac{6-5}{0.8}\right) \\ &= 1 - P(Z \leq 1.25) \qquad \text{with } Z \sim N(0,1) \\ &= 1 - 0.8944 = 0.1056 \qquad \text{using table for standard normal distribution.} \end{aligned}$$

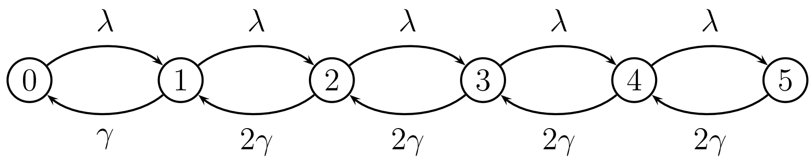
Therewith, ca. 10.56% of the cars have to be re-manufactured. Now, we like to calculate a σ that ensures that no more than 1 percent of the cars are re-manufactured.

$$\begin{aligned} & P(X > 6) \leq 0.01 \\ \Leftrightarrow\ & P\left(\frac{X-5}{\sigma} > \frac{6-5}{\sigma}\right) = P\left(Z > \frac{1}{\sigma}\right) \leq 0.01. \end{aligned}$$

From the collection of formulas we have $P(Z > 2.326) = 0.01$ and thus $2.326 \geq \frac{1}{\sigma}$. Therefore, $\sigma \leq 0.43$ ensures that no more than 1% of the cars have to be re-manufactured.

7.a

The transition graph with attached specific transition rates is given by:



In order to find the steady state equations we have to balance the rate out and the rate in in each state, i.e.,

$$\begin{aligned}
0: &\quad & \pi_1\gamma &= \pi_0\lambda & & \\
1: &\quad & \pi_0\lambda + \pi_2 2\gamma &= \pi_1(\lambda+\gamma) & &\Leftrightarrow \pi_2 2\gamma = \pi_1\lambda \\
2: &\quad & \pi_1\lambda + \pi_3 2\gamma &= \pi_2(\lambda+2\gamma) & &\Leftrightarrow \pi_3 2\gamma = \pi_2\lambda \\
3: &\quad & \pi_2\lambda + \pi_4 2\gamma &= \pi_3(\lambda+2\gamma) & &\Leftrightarrow \pi_4 2\gamma = \pi_3\lambda \\
4: &\quad & \pi_3\lambda + \pi_5 2\gamma &= \pi_4(\lambda+2\gamma) & &\Leftrightarrow \pi_5 2\gamma = \pi_4\lambda \\
5: &\quad & \pi_4\lambda &= \pi_5(2\gamma) & &
\end{aligned}$$

From the first equation we get $\pi_1 = \frac{\lambda}{\gamma}\pi_0$. Combining the two first equations gives $\pi_2 2\gamma = \pi_1\lambda$ which again give $\pi_2 = \frac{\lambda}{2\gamma}\pi_1 = \frac{\lambda^2}{2\gamma^2}\pi_0$. Inserting $\pi_1\lambda = \pi_2 2\gamma$ in the third equation give $\pi_3 = \frac{\lambda}{2\gamma}\pi_2 = \frac{\lambda^3}{2^2\gamma^3}\pi_0 = 2\left(\frac{\lambda}{2\gamma}\right)^3\pi_0$. This continues with the same structure and we generally have:

$$\pi_k = 2\left(\frac{\lambda}{2\gamma}\right)^k \pi_0 = 2\rho^k\pi_0, \quad k = 1,2,3,4,5$$

with $\rho = \frac{\lambda}{2\gamma} > 0$. Combining this with $\sum_{k=0}^5 \pi_k = 1$ and using $\sum_{k=0}^n a^k = \frac{1-a^{n+1}}{1-a}$ give:

$$\begin{aligned}
& & \pi_0 + \sum_{k=1}^{5} 2\rho^k \pi_0 &= 1 \\
&\Leftrightarrow & \pi_0\left(1 + \sum_{k=1}^{5} 2\rho^k\right) &= 1 \\
&\Leftrightarrow & \pi_0 &= \frac{1}{1 + 2\sum_{k=1}^5 \rho^k} = \frac{1}{1 - 2 + 2\sum_{k=0}^5 \rho^k} \\
& & &= \frac{1}{-1 + 2\frac{1-\rho^6}{1-\rho}} = \frac{1}{-\frac{1-\rho}{1-\rho} + 2\frac{1-\rho^6}{1-\rho}} \\
& & &= \frac{1}{\frac{-1+\rho+2-2\rho^6}{1-\rho}} = \frac{1-\rho}{1+\rho-2\rho^6}.
\end{aligned}$$

Thus $\pi_k = 2\rho^k \frac{1-\rho}{1+\rho-2\rho^6}$ for $k = 1, 2, 3, 4, 5$.
Let K denote the number of vehicles. The mean number of vehicles at the station is given by

$$E(K) = \sum_{k=0}^{5} kP(K = k) = \sum_{k=0}^{5} k\pi_k = \sum_{k=1}^{5} k\pi_k$$
$$= 2\pi_0 \sum_{k=1}^{5} k\rho^k = 2\frac{1-\rho}{1+\rho-2\rho^6}(\rho + 2\rho^2 + 3\rho^3 + 4\rho^4 + 5\rho^5).$$

Using $\rho = \frac{\lambda}{2\gamma} = \frac{0.4}{2 \cdot 0.25} = 0.8$, we get $E(K) = 2.16$.