University of Stavanger

EXAM IN: STA500 INTRODUCTION TO PROBABILITY AND STATISTICS 2

DURATION: 4 HOURS DATE: Feb 13th, 2018

PERMITTED AIDS: Approved simple calculator (HP30S, Casio FX82, TI-30, Citizen SR-270X, Texas BA II Plus or HP17bII+).

THE EXAM CONSISTS OF 5 PROBLEMS ON 3 PAGES, 18 PAGES OF ENCLOSURES.

COURSE RESPONSIBLE: Tore Selland Kleppe PHONE: 959 22 364

Note: Throughout this exam, all logarithms are natural logarithms, so that $\log(x) = \ln(x)$, and 10-based logarithms are not used.

Problem 1: Consider a bivariate distribution for (X, Y) with density

$$f_{X,Y}(x, y) = C \exp\left(-ax - by - (x + y)\right), \ x > 0, y > 0, C > 0, \ a > 0, \ b > 0.$$

a) Find the normalising constant C expressed in terms of a and b.
Find the marginal distribution of X.
Are (X, Y) dependent or independent?

Now consider the function

$$g(x, y) = \exp\left(-\frac{x + y}{2}\right).$$

b) Find $E(g(X, Y))$, assuming (X, Y) are distributed as above.

Finally consider the function

$$u(x) = \exp(-x(1 + a)).$$

c) Find the distribution of $Z = u(X)$, assuming (X, Y) are distributed as above.

Problem 2: Suppose that we observe n life times of some electronic component $T_1, \ldots, T_n$. The life times are assumed to be iid, and we assume that each life time has density

$$f_{T_i}(t) = \frac{1}{6\exp(4r)} t^3 \exp\left(-t\exp(-r)\right),\ t > 0,\ -\infty < r < \infty,$$

where r is a parameter. Suppose first we wish to estimate r based on the random sample $T_1, \ldots, T_n$ using maximum likelihood:

a) Write down the log-likelihood function and show that the maximum likelihood estimator for r is
$$\hat{r} = \log\left(\frac{\sum_{i=1}^{n} T_i}{4n}\right).$$

b) Find an exact $(1-\alpha) \times 100\%$ confidence interval for r. Hint: you may use that $Y_i = 2\exp(-r)T_i \sim \chi^2_8$.

c) Based on the invariance principle, find the maximum likelihood estimator of the transformed parameter $\beta = \exp(r)$. Find also an exact $(1-\alpha) \times 100\%$ confidence interval for β. How is this confidence interval related to the one found in b)?

Problem 3: Consider the situation where we observe n iid life times $T_1, \ldots, T_n$ that have a Weibull distribution with shape parameter 2, namely

$$f_{T_i}(t) = 2\alpha t \exp(-\alpha t^2).$$

Moreover, we consider an exponentially distributed prior with expectation b_0 (which is the same as a gamma$(1, b_0)$) for α.

a) Write down the likelihood function and find the posterior distribution of α.

Now, suppose we have $n = 3$ observations $t_1 = 0.7,\ t_2 = 1.1,\ t_3 = 0.95$, and choose $b_0 = 1$.

b) Find the Bayes estimator and a 95% credible interval for α based on the observations.

Problem 4: Consider a Markov chain with the transition probability matrix

$$P = \begin{pmatrix} 9/10 & 1/10 & 0 & 0 \\ 4/5 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

a) Draw the transition graph for this Markov chain.
Make a list of the classes and specify which classes are transient and which are recurrent.

Now consider a different situation: A web server is requested for a particular page as a Poisson process with rate 4 per hour. Suppose you know that 34 requests arrived between 08:00 AM (i.e. 08:00 in Norwegian format) and 04:00 PM (i.e. 16:00 in Norwegian format).

b) What is the probability that the first request came before 08:15 AM (08:15)?

Problem 5: The "modulus 3" function $\text{mod}_3(x)$ is defined, for non-negative integers as

$$\begin{aligned} \text{mod}_3(0) &= 0 \\ \text{mod}_3(1) &= 1 \\ \text{mod}_3(2) &= 2 \\ \text{mod}_3(3) &= 0 \\ \text{mod}_3(4) &= 1 \\ \text{mod}_3(5) &= 2 \\ \text{mod}_3(6) &= 0 \\ \text{mod}_3(7) &= 1 \\ &\vdots \end{aligned}$$

Consider a stochastic process so that for $0 < p < 1$,

$$X_{t+1} = \begin{cases} X_t & \text{with probability } p, \\ \text{mod}_3(X_t + 1) & \text{with probability } 1 - p, \end{cases}$$

and $X_0 = 0$.

a) Argue for why this process is a Markov chain.
Draw the transition graph and write down the transition probability matrix.
Why is this process irreducible?
Why is this process aperiodic?

b) Find the steady state probabilities for this process. Hint: it is sufficient that you guess the steady state probabilities and plug them into corresponding steady state equations.

Solutions

1,a)
It is relatively straight forward to observe that

$$f(x,y) = C\exp(-x(a+1))\exp(-y(b+1))$$

which is recognised as the product of two exponential distribution kernels (in rate parameterisation). Therefore we must have that e.g. $\int \exp(-x(a+1))dx = (a+1)^{-1}$, and it follows that $C = (a+1)(b+1)$.
Using the same argument as above, it is clear that $f(x,y) = (a+1)\exp(-x(a+1)) \times (b+1)\exp(-y(b+1))$, thus X (marginally) has an exponential distribution with rate $(a+1)$, alternatively mean $1/(a+1)$.
Also following from above, it is clear that X and Y are independent as $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.
Alternatively all of these questions could have been done using variations of $\int D\exp(-cx)dx = D/c$.

1,b)
The sought expectation is

$$\begin{aligned}
E(g(X,Y)) &= \int_0^\infty \int_0^\infty \exp\left(-\frac{x+y}{2}\right) C\exp(-x(a+1)-y(b+1))dxdy \\
&= C\int_0^\infty \int_0^\infty \exp(-x(a+1+1/2))\exp(-y(b+1+1/2))dxdy \\
&= \frac{C}{(a+1+1/2)(b+1+1/2)} \\
&= \frac{(a+1)(b+1)}{(a+1+1/2)(b+1+1/2)}
\end{aligned}$$

1,c)
This is a transformation of random variables problem. The forward transformation $y = u(x)$ is monotone (decreasing), an therefore admit a unique inverse, namely

$$\begin{aligned}
y &= u(x) = \exp(-x(1+a)) \\
&\Downarrow \\
\log(y) &= -x(1+a) \\
&\Downarrow \\
-\frac{\log(y)}{(1+a)} = w(y) &= x.
\end{aligned}$$

Moreover, the derivative of $w(y)$ is $w'(y) = -(y(a+1))^{-1}$ which has absolute value $(y(a+1))^{-1}$. Finally, it was found above that X has an exponential distribution with rate parameter $a+1$. Plugging this information into the transformation formula, we obtain that

$$\begin{aligned}
g(y) = (a+1)\exp(-(a+1)w(y))|w'(y)| &= \frac{a+1}{y(a+1)}\exp\left(\frac{(a+1)\log(y)}{a+1}\right) \\
&= \frac{1}{y}\exp(\log(y)) = 1.
\end{aligned}$$

I.e. $g(Y)$ has a uniform distribution over a unit length interval. The boundaries for this uniform distribution are found by looking at

$$u(0) = 1, \ \lim_{x\to\infty} u(x) = 0$$

Thus this uniform distribution is between $(0, 1)$.

2,a)
Log likelihood: first we consider the logarithm of the density of a single observation:

$$\log(f_{T_i}(t_i)) = \text{"constant with respect to } r\text{"} - 4r - t_i \exp(-r).$$

Thus, the relevant part of the log-likelihood function is

$$l(r; T_1, \ldots, T_n) = \text{constant} - 4nr - \exp(-r) \sum_{i=1}^{n} T_i.$$

The MLE $\hat{r}$ is found by equating the first derivative of $l(r)$ with zero

$$\begin{aligned} 0 &= \frac{\partial}{\partial r}\left[-4nr - \exp(-r)\sum_{i=1}^{n} T_i\right] \\ &= -4n + \exp(-r)\sum_{i=1}^{n} T_i \\ \frac{4n}{\sum_{i=1}^{n} T_i} &= \exp(-r) \\ \hat{r} &= \log\left(\frac{\sum_{i=1}^{n} T_i}{4n}\right). \end{aligned}$$

Finally, we check that this critical point is indeed a maximum:

$$\frac{\partial^2}{\partial r^2} = -\exp(-r)\sum_{i=1}^{n} T_i < 0,$$

as $T_i > 0, \ i = 1, \ldots, n$.

2,b)
In order to find $g(\hat{r}, r)$, using the hint, we observe that

$$\exp(\hat{r}) = \frac{1}{4n}\sum_{i=1}^{n} T_i = \frac{1}{4n}\frac{\exp(r)}{2}\underbrace{\sum_{i=1}^{n} Y_i}_{\sim\chi^2_{8n}} \tag{1}$$

Thus

$$g(\hat{r}, r) = 8n\exp(\hat{r})/\exp(r) \sim \chi^2_{8n}.$$

Based on this g-function, we have that

$$\begin{aligned} 1-\alpha &= P\left(\chi^2_{8n,1-\alpha/2} < 8n\exp(\hat{r})/\exp(r) < \chi^2_{8n,\alpha/2}\right) \\ &= P\left(\frac{1}{\chi^2_{8n,\alpha/2}} < \exp(r)/(8n\exp(\hat{r})) < \frac{1}{\chi^2_{8n,1-\alpha/2}}\right) \\ &= P\left(\hat{r} + \log\left(\frac{8n}{\chi^2_{8n,\alpha/2}}\right) < r < \hat{r} + \log\left(\frac{8n}{\chi^2_{8n,1-\alpha/2}}\right)\right) \end{aligned}$$

which gives the CI

$$\left[\hat{r} + \log\left(\frac{8n}{\chi^2_{8n,\alpha/2}}\right), \hat{r} + \log\left(\frac{8n}{\chi^2_{8n,1-\alpha/2}}\right)\right].$$

2 c)
Due to the invariance principle, the maximum likelihood estimator for $\beta = \exp(r)$ is simply

$$\hat{\beta} = \exp(\hat{r}) = \frac{\sum_{i=1}^n T_i}{4n}.$$

To find the CI, we take as vantage point (1):

$$\exp(\hat{r}) = \hat{\beta} = \frac{\sum_{i=1}^n T_i}{4n} = \frac{\beta}{8n} \underbrace{\sum_{i=1}^n Y_i}_{\sim \chi^2_{8n}},$$

thus

$$g(\hat{\beta}, \beta) = \frac{8n}{\beta}\hat{\beta} \sim \chi^2_{8n}.$$

This leads to, via similar arguments as above:

$$\begin{aligned} 1 - \alpha &= P\left(\chi^2_{8n,1-\alpha/2} < 8n\hat{\beta}/\beta < \chi^2_{8n,\alpha/2}\right) \\ &= P\left(1/\chi^2_{8n,\alpha/2} < \beta/(8n\hat{\beta}) < 1/\chi^2_{8n,1-\alpha/2}\right) \\ &= P\left(8n\hat{\beta}/\chi^2_{8n,\alpha/2} < \beta < 8n\hat{\beta}/\chi^2_{8n,1-\alpha/2}\right) \end{aligned}$$

which gives the CI:

$$\left[8n\hat{\beta}/\chi^2_{8n,\alpha/2}, 8n\hat{\beta}/\chi^2_{8n,1-\alpha/2}\right]$$

Note that this CI could have been obtained by simply applying the exponential function to the CI in the original parameterization, e.g. left hand limit:

$$\exp\left(\hat{r} + \log\left(\frac{8n}{\chi^2_{8n,\alpha/2}}\right)\right) = \exp(\hat{r})\frac{8n}{\chi^2_{8n,\alpha/2}} = \hat{\beta}\frac{8n}{\chi^2_{8n,\alpha/2}}$$

This is an example of the fact that we may also apply the invariance principle in the case where the applied transformation is monotone.
3,a)
Likelihood function:

$$L(\alpha; T_1, \ldots, T_n) = \prod_{i=1}^n 2\alpha T_i \exp(-\alpha T_i^2) \propto \alpha^n \exp\left(-\alpha \sum_{i=1}^n T_i^2\right)$$

Posterior kernel:

$$p(\alpha | T_1, \ldots, T_n) \propto \alpha^n \exp\left(-\alpha \sum_{i=1}^n T_i^2\right) \exp(-\alpha/b_0) = \alpha^{(n+1)-1} \exp\left(-\alpha\left(\sum_{i=1}^n T_i^2 + \frac{1}{b_0}\right)\right)$$

which is recognized to be Gamma kernel with shape parameter $a = n + 1$ and scale parameter $b = \left(\sum_{i=1}^{n} T_i^2 + \frac{1}{b_0}\right)^{-1}$.

3,b)
First, we find the parameters $a = n + 1 = 4$ and

$$b = \frac{1}{0.7^2 + 1.1^2 + 0.95^2 + 1} = 0.277585$$

Thus, the Bayes estimator (posterior mean) is $ab = 1.11034$. The credible interval is constructed by first observing that

$$\frac{2\alpha}{b} \sim \chi_8^2.$$

Thus,

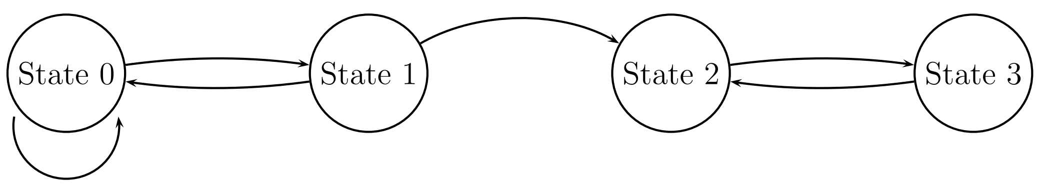
$$0.95 = P\left(\chi^2_{0.975,8} < \frac{2\alpha}{b} < \chi^2_{0.025,8}\right) \quad (2)$$

$$= P\left((b/2)\chi^2_{0.975,8} < \alpha < (b/2)\chi^2_{0.025,8}\right). \quad (3)$$

From the table, we have that $\chi^2_{0.975,8} = 2.180$ and $\chi^2_{0.025,8} = 17.535$, and therefore we obtain the credible interval

$$[0.5 \times 0.277585 \times 2.180, 0.5 \times 0.277585 \times 17.535] = [0.3025, 2.433].$$

4,a)
We start by drawing the transition graph:



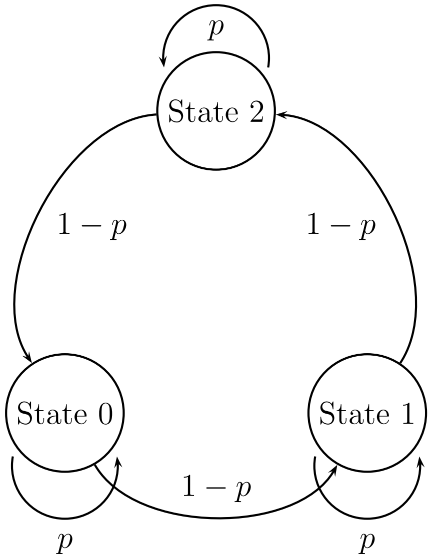
The Markov process has classes $\{0, 1\}$ and $\{2, 3\}$ where the former is transient and the latter is recurrent.

4,b)
We know that 34 request came in a period of $8 \times 60 = 480$ minutes, and we are asked whether at least one request happend in the first 15 minutes. Let $X = \#$ of requests in first 15 minutes. Then

$$X \sim \text{Binomial}(34, 15/480)$$

and we are asked for $P(X \geq 1) = 1 - P(X = 0) = 1 - (1 - 15/480)^{34} = 0.66022$.

5,a)
The process is a Markov chain as the process has a discrete state space $0, 1, 2$, and the distribution of $X_{t+1}|X_t, X_{t-1}, \ldots$ depends only on X_t and not on earlier values. The transition graph is given as:



The transition probability matrix is given as:

$$P = \begin{pmatrix} p & 1-p & 0 \\ 0 & p & 1-p \\ 1-p & 0 & p \end{pmatrix}.$$

As p is assumed to be < 1, the chain is irreducible, i.e. one class. As p is assumed to be > 0, the process is aperiodic (e.g. all of the diagonal elements of P are non-zero).

5,b)
As the graph is invariant to renaming of the nodes by addition mod 3, it is clear that the steady state probabilities must be uniform, i.e. $pi_0 = \pi_1 = \pi_2 = 1/3$. To verify this, we consider first $\pi_0 + \pi_1 + \pi_2 = 3/3 = 1$, and finally the equations corresponding to the two first columns of P:

$$p\pi_0 + (1-p)\pi_2 = (p + (1-p))1/3 = 1/3 = \pi_0.$$

$$(1-p)\pi_0 + p\pi_1 = ((1-p) + p)1/3 = 1/3 = \pi_1.$$

This shows that the uniform distribution solves the steady state equations.

6,a)